REVIEW

Advanced turbulence closure models: a view of current status and future prospects

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This report presents the status and reviews some of the current activities in the development of advanced one-point turbulence closure models with special reference to the modeling and computation of turbulent wall flows. It gives a brief retrospective of the development of the one-point closure models and highlights major achievements. It also recalls some of the pertinent issues that are still open and that pose challenges to modelers. Some pertaining deficiencies, which have been obvious since the very beginning of the development and are still present, are discussed, together with some marked differences in the views of various groups of modelers.

The review is limited to conventional one-point two-equation Eddy Viscosity Models (EVMs) and Differential Second-Moment (Re-stress) Models (DSMs). It does not deal with any spectral approach, nor partial field models, such as subgrid modeling or similar methods used within the framework of other simulation techniques. Some new approaches, such as the Renormalization group theory (RNG), are mentioned briefly and only insofar as they may inspire modifications and improvements of conventional models.

The report deals with general aspects of modeling complex turbulent flows, but is restricted to incompressible (or mildly compressible) fluid, with particular focus on computation of wall flows.

Keywords: turbulence models; second-moment closure models; turbulent wall flows; low-Re number Re-stress models

1. Introduction

Ever since the first successes, just a quarter of a century ago, in reproducing computationally the experimental data on field properties in a wide range of turbulent flows with a single set of modeled equations and empirical coefficients, one-point closure models have rapidly gained in popularity. Turbulence modeling for computational fluid dynamics emerged as a distinct discipline and, in conjunction with suitable numerical methods, became the most widely employed predictive tools in fluid mechanics and in heat and mass transfer. Numerous researchers and users all over the world contributed to the testing, development, and refinement of these tools.

In spite of a wide recognition of the general achievements in mathematical modeling of turbulence over the past 20 years, there are still some physical effects the modeling of which has not yet resulted in a comparable feeling of satisfaction in the turbulence research community. Streamline curvature and extra strain rate, unsteadiness and periodicity, viscosity and wall-proximity, three-dimensionality, flow separation and reversal, buoyancy and rotation, etc., are perhaps the most prominent examples. These effects are often present in many turbulent flows. Yet their modeling still poses difficulties, not only because of an insufficient understanding of the physics, but also because of an apparent need to employ mathematical formulations and numerical schemes of greater complexity than had been hitherto used to model simpler, well-behaved flows. Various attempts to incorporate some or all of the above-mentioned effects and to devise a general turbulence model have produced only partial successes, usually at the expense of physical clarity and computing economy. Such a discouraging outcome has caused some of the proposed concepts to become obsolete, but the necessity for proper accounting of these effects prompts a continuation of the search for better models. Some recent developments in experimental techniques, in particular the laser Doppler anemometry, particle-imaging technique, and holography, produce more and more new experimental evidence on some turbulence interactions that has not been possible to measure accurately hitherto. A particularly fresh impetus came with advances in direct numerical simulation of turbulence, which offers new possibilities of obtaining information and of verifying the hypotheses on some turbulence properties that are still inaccessible by the presently available experimental techniques.

While efforts at improving and refining models are continuing, in particular at a few reputed schools, numerous engineers all over the world have succumbed to the appeal of an attractive blend of simplicity and acceptable predictive ability of popular models, such as k–ε, and have employed them for the computation and predictions of turbulent flows and associated transport phenomena in very complex geometries. Some of these results indeed look very impressive, with predicted details of velocity field, turbulence properties and

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other variables that were inconceivable only a few years ago. However, Lumley's (1978) warning, issued a decade and a half ago, that the application of a technique in situations in which data do not exist "must be regarded as a dangerous practice since the limitations of the technique are not known with any precision," is still as valid as ever. Although we now know much more, active researchers feel that a dose of caution is still often advisable and that not all of the computed results are to be fully trusted. A simple turbulence model may be incapable of capturing certain interactions when applied to what Reynolds (1976) called an "exotic" flow geometry, simply because such interactions may have been unimportant in previously considered flows. Inadequate specification of boundary conditions, and/or numerical defects of which the user may be unaware, may yield very misleading results and the subsequent abuse of these findings. A minimum of both modeling and computing expertise is a prerequisite for trustworthy results.

However, in spite of all these cautionary notes, prediction results can often serve engineering purposes, simply because the uncertainties in modeling other phenomena involved may be much higher than those implied in the model of flow hydrodynamics. Or, as Bradshaw (1987) wrote in his review of secondary flows, "the best modern methods allow almost all flows to be calculated to higher accuracy than the best-informed guess, which means that the methods are genuinely useful even if they cannot replace experiments."

2. A few historical recollections

The topic of turbulence modeling and computation has been very popular since its first appearance, and a number of authors have taken up the challenge in the past to write review articles (e.g., Bradshaw 1972; Mellor and Herring 1973; Reynolds 1976; Lumley 1978; Lakshminarayana 1986; Nallasamy 1987; Rodi 1982, 1986, 1988; Launder 1990; So et al. 1991; Speziale 1991; and others). However, over the past few years a number of new publications have reported progress in refining turbulence models, inspired mainly by the results of direct numerical simulation. Some new approaches based on more rigorous mathematical derivation, such as the Renormalization group theory (RNG) (e.g., Yakhot and Orszag 1986; Yakhot et al. 1992; Yakhot and Smith 1992) or the Elliptic relaxation method of Durbin (1991, 1993), claim major breakthroughs in turbulence simulation. These new developments, after a longer period of stagnation, came toward the end of 1993, which marked a quarter of a century after the (we may say historic) 1968 AFOSR-IFP Stanford Conference on Computation of Turbulent Boundary Layers. In the words of J. Lumley (1978), this conference "was the birth of the technique which has become known as second order modelling ...."

Of course, the 1968 Stanford meeting was not the beginning, but only the turning point that legitimized the superiority of the mean-field approach to integral methods and stimulated their further development. As is well known, the work started much earlier with Kolmogorov (1941), Chou (1945), Rotta (1951), and Davidov (1959, 1961), whose ideas materialized in practical computation only after the advent of high-speed computers. Already in the year of the Stanford meeting, frantic research was going on in parallel at a few places in the world. At the Los Alamos Laboratory, Harlow and Nakayama (1968) and Daly and Harlow (1970) were working on the development of a "generalized transport theory of anisotropic turbulence" employing the partial differential equations for turbulent stresses and energy decay rate following earlier suggestions of Chou (1945). Donaldson et al. (1972) pursued invariant modeling to close the Re-stress equations by employing a prescribed length scale. At the Imperial College, the successful development of the Patankar–Spalding (1968) numerical code for solving the partial differential equations prompted a rapid development of turbulence models. Ng, Rodi, and Spalding (see, e.g., Rodi and Spalding 1970) pursued the k–kL type of model, while Hanjalić, Jones, and Launder (1970) preferred to investigate the two- and three-equation models with energy dissipation rate ε as the scale parameter, which even at that time showed some practical advantages that became generally recognized some years later. A number of parabolic wall and free flows were solved with a single set of equations and empirical coefficients that have changed little since the work of Hanjalić and Launder (1972). The k–ε model was already extended to be applicable to low-Re-number flows (Jones and Launder 1972).

However, apart from some of the latest developments such as the RNG theory, which must await further scrutiny before they are widely accepted, nothing substantial has really changed from those original formulations of two-equation models. The intensive research efforts over the past two-and-a-half decades have, however, expanded the frontiers of our knowledge and resulted in numerous proposals for models upgrading and improvements. A rich experience on the extent and limitations of predictive abilities of these and higher-order models has been accumulated by the enlarged turbulence-modeling community. The 1980–81 AFOSR-HTTM Conference on Complex Turbulent Flows, envisaged at that time as a "decennary grand festival of achievements in turbulence modelling and computation," attracted about 40 research teams who tested their models against a number out of 50 predefined test cases for which independent experimental data were supplied. Most of the models demonstrated at that time a high potential and a scope of applicability, with prospects of further improvements and wider use in computation of complex turbulent flows of industrial and environmental relevance. The end of the last decade called for a new review of the state of the art, and an international project run from Stanford was launched, "Collaborative testing of turbulence models" 1990–1992 (Bradshaw et al. 1991), which was envisaged to use the modern methods of postal communication instead of personal attendance at a conference. New, challenging test cases were proposed, most of them substantiated this time by results of direct numerical simulation. Partly due to lack of live communication and partly due to saturation and a slowdown in the research, this project did not bring many novelties: most participants presented data with the same or very similar rudimentary models used a decade earlier. In spite of this, the consensus among the modelers, using even the same or very similar models, was below expectation. The project was a success at least in the sense that it emphasized very many deficiencies and weaknesses of contemporary turbulence models. Reynolds-stress models appeared to have an advantage over two-equation eddy-viscosity models, but not by a convincing margin.

3. Current engineering models

Three types of turbulence models could be identified as *fast* engineering methods (despite the need to solve partial differential equations):[1]

- two-equation eddy-viscosity models (EVMs);
- the differential Re-stress equation model (DSM) (differential second-moment closure); and
- intermediate (truncated) and "hybrid" models that hierarchically fall in between these two and take some advantages from each of them:
  - — Algebraic stress models (ASMs); and
  - — Incomplete or partial stress models (PSMs)

Although the foundations of *two-equation models (EVMs)* are well known, in a search for model refinements it is helpful to reiterate the basic assumptions:

- turbulent stresses are expressed as directly proportional to the mean rate of strain;
- the proportionality coefficient in the stress–strain relationship—the eddy viscosity—is expressed in terms of two parameters, which can be grouped so as to represent a characteristic turbulence scale;
- the two parameters are obtained from modeled differential transport equations, which are based upon (or, at least, have some roots in) the exact transport equations for these quantities, which take into account the time, memory, and spatial awareness of the local turbulence state; and
- basic two-equation EVMs presume a linear relationship between the turbulent stresses and mean rate of strain and treat the eddy viscosity as a scalar property of the flow.

Two-equation EVMs in their rudimentary forms have a major advantage in their simplicity and practical usability. A computational advantage (but also a physical deficiency!) is the scalar (isotropic) eddy viscosity, which allows the model to be incorporated into any existing laminar Navier–Stokes computer code.

The differential Re-stress model (DSM) is regarded as the natural and most logical level of modeling within the framework of the Reynolds averaging approach, since it provides the extra turbulent momentum fluxes from the solution of full transport equations, which are derivable from the Navier–Stokes equations. In addition to the solution of equations for each Re-stress component, it requires a length-scale-supplying equation, for which the great majority of models employ the dissipation rate equation. In such a form, the DSM model still represents the most comprehensive description of turbulent flows that can be employed for practical computations with the present generation of computers.

In addition to a more faithful description of the turbulence dynamics (e.g., both the convection and diffusion are accounted for in a differential form), second-moment closure models are legitimized by a stricter compliance with the general modeling principles on which some modelers particularly insist. These principles can be grouped into two classes: the first class can be regarded as a mathematical formalism, and it implies the dimensional coherence of all terms in equations, tensorial-order consistency, and coordinate-frame and material-frame indifference, as well as satisfaction of realizability conditions, limiting properties of two-dimensional (2-D) turbulence, etc. The second class of principles is of physical nature and is more difficult to quantify. These can be defined as principles of physical coherence that postulate, e.g., that the turbulence correlations be primarily modeled in terms of turbulence parameters that are known to govern directly the described interactions, instead of mean-flow or some external parameters. The decreasing influence of higher-order moments upon the mean-flow properties, confirmed in many flows of practical relevance, can be regarded as one of the principles of this type.

As is well known, a more general compliance with these and other rules necessarily burdens models with more complex mathematical formalism (Mjolsness (1979) proposed no less than eight rules, including the principle of super-realizability!), which for many practical flow situations may not produce any qualitative improvements of predictions. For this reason, there is no general consent with respect to the strict obedience of the modeling principles. Disobedience causes discomfort to some modelers, while others, more technologically oriented, tend to sacrifice mathematical puritanism for the sake of practicality. The question arises, however, whether compliance with a wider set of constraints ensures a higher degree of model universality. This does seem to be generally the case, and with further advances of computers we may expect more principles of invariance and realizability to be implemented in models of all levels.

Hybrid methods should be regarded as a compromise of today. In most cases they will be used by knowledgeable workers who know enough about the affair to be able to estimate to what extent they can prune the full transport equations to achieve computational advantages, but still gain benefits beyond those offered by the simple two-equation models.

Some weaknesses and shortcomings that have been known from the early days of models and have persisted as still unresolved will be listed briefly. For *two-equation EVMs*, these are

- linear (Newtonian) stress–strain relationship through the eddy-viscosity model (at least in the rudimentary linear models);
- scalar (isotropic) character of eddy viscosity (insensitivity to the orientation of the turbulence structure and its transporting and mixing mechanisms);
- inability to reproduce stress anisotropy and its consequence (e.g., prediction of stress-induced secondary motion);
- scalar character of turbulence scales—insensitivity to eddy anisotropy;
- limitations to define only one time- or length scale of turbulence for characterizing all turbulence interactions;
- failure to account for all physical processes governing the behavior of ε or other scale-determining quantities by virtue of the simplistic form of the basic equation for that variable;
- inadequate incorporation of viscosity damping effects on turbulence structure (low Re-number models);
- inability to mimic the preferentially oriented and geometry-dependent effects of pressure reflection and eddy-flattening and squeezing mechanisms due to the proximity of solid- or interphase surface; and
- frequently inadequate treatment of boundary conditions, in particular at the solid wall.

Differential Re-stress models overcome the first three deficiencies, but all the others remain to a greater or lesser extent. In addition, this class of models involves some further uncertainties, such as

- deficiencies in modeling various terms in $\overline{u_i u_j}$-differential equations (in particular the pressure redistribution, but also the turbulent diffusion and stress dissipation rates).

The modeling of various turbulence interactions, represented by different terms in transport equations, requires a specification of characteristic turbulence time-, length- and velocity scales. Of the two turbulence properties used to define characteristic scales, the kinetic energy of turbulent fluctuations $k = \overline{u_i u_i}/2$ has been employed without exception, since it has proved to be the best defined and most readily obtainable turbulence parameter. As the second turbulence property, the most popular has been the homogeneous part of the rate of energy dissipation $\varepsilon = \nu\overline{(\delta u_i/\delta x_j)^2}$, which in combination with k produces the turbulence time- and length scale. Other variables have also been persistently used by other researchers, such as a length scale L and the product (kL) (e.g., Rotta 1951; Mellor and Herring 1973), or $\tilde{\omega}^2$—the mean square of the fluctuating vorticity—or its *rms*, $\tilde{\omega}$ ($\omega_k = \epsilon_{ijk}\delta u_i/\delta x_j$) (e.g., Wilcox 1988; Wilcox and Traci 1976; Wilson and Rubesin 1980). The latter variable can be interpreted as the characteristic frequency of the large eddy structure, or as the specific dissipation rate per unit of kinetic energy $\tilde{\omega} = \varepsilon/k$. More recently, the turbulence time scale τ (reciprocal of the eddy frequency), which reduces to the ratio k/ε in the single-point models, was briefly popular and looked like a promising new scale-supplying variable. Like the length scale, it appeals more than other variables because of its plausible physical meaning, and also its plausible behavior in many flow regions, particularly in the wall vicinity. However, the verifications reported so far have not brought many improvements nor specific advantages in comparison with existing variables, particularly ε (e.g., Speziale et al. 1992; Thangam et al. 1992). The major deficiency seems to be in the rigidity of the tested τ-equation, which, being derived from the current k and ε equations for high Re-number flows, contains a constant sink term.

The above-mentioned (and other) scale-supplying variables can be expressed as a product $k^m\varepsilon^n$. Irrespective of the choice of m and n, a transport equation for the scale-supplying variable can easily be derived empirically (although an origin can be traced to the Navier–Stokes equations) that has the same conservation form with source and transport terms. The choice is more a matter of taste, since all variables and their equations are derivable one from another, differing only in the way the diffusion terms are modeled. The greater popularity of the dissipation equation can be attributed to the practical advantage of its simpler form as compared with other equations. It should be noted that in most cases, other than when using ε, it appears necessary to retain a cross-diffusion term like $(\partial k/\partial x_j)(\partial\varepsilon/\partial x_j)$, which is not of diffusive character and should be treated as a source.

The ε-equation has a form of the simplest transport equations, with terms representing the basic processes that govern its dynamics. Despite its striking simplicity, the basic equation for the turbulence energy dissipation has proved to possess a surprising level of generality, which explains its popularity as the length-scale-supplying equation. Extensive tests have revealed, however, a number of shortcomings that appear particularly in flows with a pronounced nonequilibrium of basic turbulence mechanisms.

3.1. Estimate of performances

The earliest comprehensive survey of the performance of various models can be found in the *Proceedings of the 1980–81 AFOSR-HTTM Stanford Conference on Complex Turbulent Flows* (Kline et al. 1981). More recently, several up-to-date reviews have appeared covering more specific aspects: Lakshminarayana (1986) surveyed practices of modeling the curvature, rotation, and three-dimensionality, Cousteix (1986) reviewed modeling and computation of three-dimensional and unsteady boundary layers, Bradshaw (1987) gave an interesting account of current models' ability to predict secondary flows of various origins, while Nallasamy (1987) gave a review of the application of turbulence models to the prediction of internal flows. A general report on the EUROMECH 180 Colloquium on turbulence modeling for incompressible flows by Rodi (1986) gave a good review of the state of the art as seen by the Colloquium participants. Other interesting readings include more recent general reviews by Rodi (1988), Launder (1990), So et al. (1991), Speziale (1991), and others.

Although the comments and conclusions in these reviews reflect personal convictions and some of them may not hold generally, most of them are shared overwhelmingly by the majority of active members of the turbulence-modeling community, since these conclusions derive from the experience of a vast number of current users. It would be beyond the scope of the present review's objective and an unnecessary repetition to summarize conclusions, but some may be recalled as a reminder for the forthcoming discussion (limited to incompressible flows).

- *Two-equation eddy viscosity models (EVMs)* in their basic form yield satisfactory predictions of 2-D thin shear flows (attached boundary layers and 2-D flows in conduits, with low-to-moderate pressure gradients, wall suction and blowing, free flows), as well as some recirculating flows that are dominated by pressure gradients, and even some flows with streamline curvature and body force when these effects are weak.
- Two-dimensional flows with separation and stronger curvature, rotation, buoyancy and other specific effects can be predicted satisfactorily with two-equation models *if* adequate modifications are made. The simplest, though not always effective, way is to make the coefficient in the eddy viscosity, C_μ, a function of the P/ε ratio as well as of extra effects (preferably derived from a Reynolds-stress model and expressed in forms of the nondimensional similarity parameters: the Richardson, Rossby, and Rayleigh numbers). A still sounder approach implies the inclusion of additional terms into the dissipation equation, preferably based on the exact terms in the original equations. However, most of the modifications proved to be problem dependent and their effects restricted only to the class of flows for which they were designed.
- *Algebraic stress models (ASMs)* that are derived from parent differential stress equations have been reported to yield better predictions of 2-D flows with secondary motion and in some flows with rotation and curvature, provided that these flows are not far from equilibrium, i.e., in absence of rapid change of the mean rate of strain and if the stress transport is of minor importance. Although the ASMs by their origin are supposed to capture better flow physics, e.g., the stress anisotropy, streamline curvatures, and body forces, their only indisputable merit in comparison with the EVMs is their ability to reproduce some kind of secondary motion in noncircular ducts and corners.
- *The differential Re-stress models (DSMs)* yield superior predictions of 2-D nonequilibrium flows (sudden and strong streamwise variations of boundary and external conditions, in particular those imposing the abrupt changes of strain rate). They also perform better in unsteady and periodic flows and everywhere where the response of turbulence field exhibits a lag and a consequent hysteresis as compared with the variation of mean flow properties. DSMs account automatically for the effects of stress anisotropy (e.g., stress-induced secondary flows), streamline curvature, and flow rotation (formally also for three-dimensionality) and perform usually better where these effects are important. Of course, shortcomings in the scale-determining equation, which becomes particularly important in pressure-dominated flows, or inadequate treatment of boundary conditions, like the use of wall functions, often annul or diminish the inherent superiority of the DSMs as compared with EVMs, leading to a wrong judgment.

An increased interest in the application of turbulence models to the calculation of more complex 2-D and, particularly, three-dimensional (3-D) flows over the past few years, revealed additional shortcomings of the present models at all levels. For example, Craft and Launder (1991) found that the popular model of the wall reflection on the stress redistribution due to the pressure reflection, designed originally for wall-parallel flows, produces in the stagnation region of a jet impinging on a surface an opposite effect from that detected by experiments. Bradshaw (1987) argued that none of the models in their present forms can *fully* capture the significant changes in turbulence structure imposed by even mild three-dimensionalities. Indeed, the physical understanding of the effects of strong skew-induced vorticity, embedded vortices, and even stress-induced vortices upon the turbulence structure is regarded as still insufficient (partly due to the lack of experimental data) to suggest plausible model extensions (e.g., the influence on the pressure-redistributing process). Some recent predictions with standard models show an interesting outcome (e.g., Liandrat et al. 1987, Figure 4). Schwarz and Bradshaw (1992) analyzed several models of turbulent diffusion and of pressure strain terms on the basis of experimental data in a 3-D boundary layer in a curved channel and concluded that even the basic Re-stress model can yield plausible predictions in some simple 3-D flows. These cases are, however, more exceptions than rules, and reliable solutions of complex 3-D flows, particularly those with flow separation, still remain a major task and an exciting challenge. Major weaknesses of models at all levels are still to be discovered. Certain facts, though, are already more than obvious: basic EVMs with isotropic eddy viscosity are totally inadequate for 3-D flows, as illustrated in Figures 1 and 2, where the eddy viscosities for two directions are presented for two relatively simple 3-D flows.

Computationally, of course, two-equation models offer frequently decisive advantages over the DSMs models. Almost the same could be claimed for the ASMs models, provided simpler geometries are considered. However, these models are often shown to pose numerical instabilities or other kinds of computational inconveniences, and the question arises whether the direct use of full or pruned differential Re-stress models would be a sounder approach in general, provided that adequate computing facilities are available. Also, for irregular flow domains, where the use of nonorthogonal coordinate systems is required, and for 3-D flows, the ASMs expressions become cumbersome and their advantages over the DSMs models diminish.

The above are more general statements with which some meticulous modelers would not agree, since there is evidence of many yet unresolved paradoxes that conflict with the above conclusions—for example, the unsatisfactory predictions of the rate of spread of round jets, the inability to predict the weakly swirling jet (contrary to the case of the strongly swirling jet, which can sometimes be satisfactorily predicted even by the standard k–ε model), etc.

Controversies arise sometimes from uncertainties and inconsistencies in the definition of inflow turbulence quantities and in the treatment of boundary conditions (whether wall

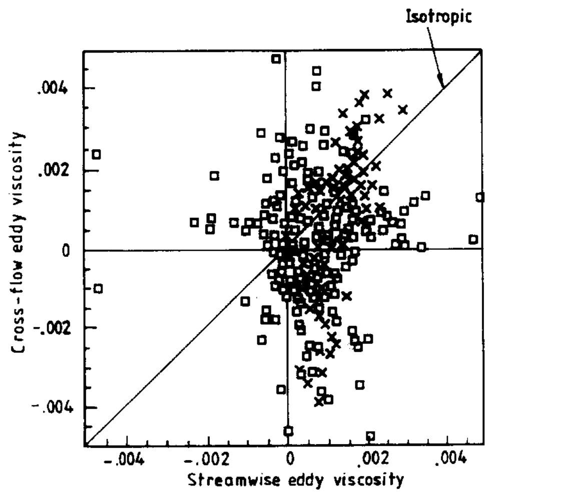

Figure 1 Spanwise vs. streamwise eddy viscosity in local mean flow coordinates inside a 3-D boundary layer (□—points inside the line of separation) (Davenport and Simpson 1990)

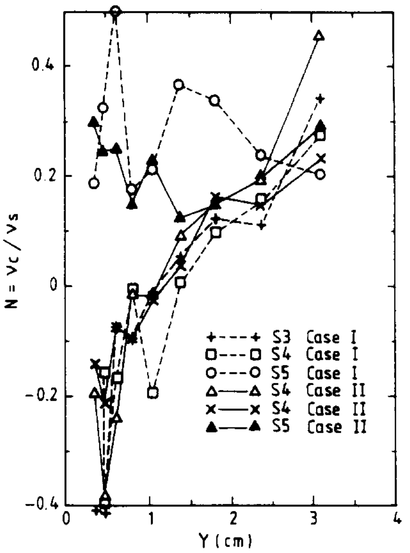

Figure 2 Eddy-viscosity ratio on selected streamline in a pressure-driven 3-D boundary layer created by an upstream-facing wedge (Anderson and Eaton 1987)

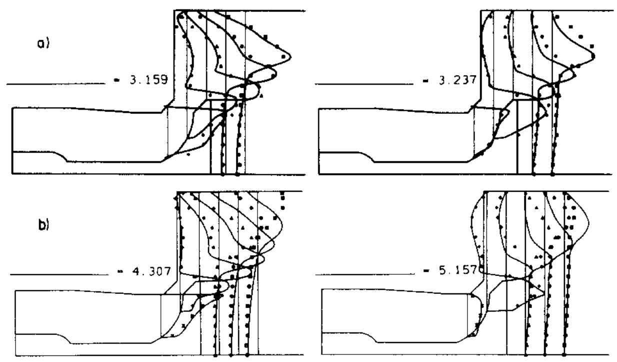

Figure 3 Axial (left) and radial (right) velocities around a poppet valve: (a) 10-mm valve lift; (b) 6-mm valve lift (Lilek et al. 1991)

functions or a straight-to-the-wall integration is applied), as well as from insufficient accuracy of the applied numerical schemes; all these, as well as insufficient testing of the model with varying boundary conditions, may obscure the real performances of the turbulence models and, hence, lead to misjudgments. An example of this kind is shown in Figure 3 (Lilek et al. 1991), where the k–ε solutions of the flow around a poppet valve for two valve openings are compared with measurements. Whereas for a 10-mm valve lift, both the axial and radial velocities are in good agreement with experiments at all considered cross sections (Figure 3a), the solutions for a 6-mm valve lift depart substantially from the measurements.

Figures 4 to 7 illustrate by way of a few selected examples the performances of the basic Re-stress models in several simple flows with some specific features. For comparison, Figures 5 to 7 show also the k–ε solutions, which are visibly inferior. More illustrations in support of the DSM, particularly in flows with extra strain rates, can be found elsewhere (e.g., Launder 1990; Leschziner 1989). Needless to say, numerous examples can also be found in the literature where the DSM yields no decisive improvement in comparison with EVMs.

Figure 5 compares the DSM computations and measurements of some properties of a turbulent boundary layer subjected to periodic disturbances (Hanjalić and Stošić 1985).

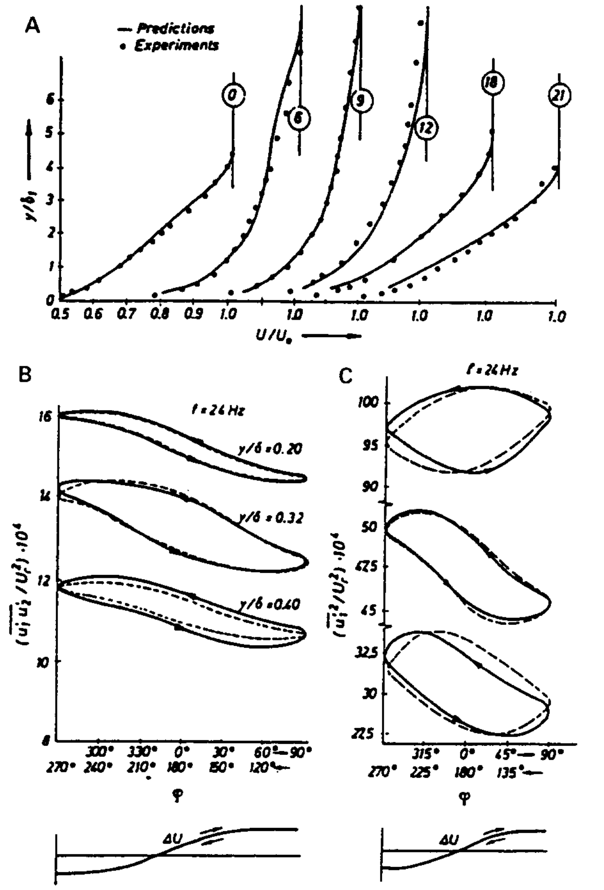

Figure 4 Hysteresis of turbulent shear stress at different flow depths in a pulsating boundary layer. ---: Experiments (Acharya and Reynolds 1975); ——: Computations (Hanjalić and Stošić 1985)

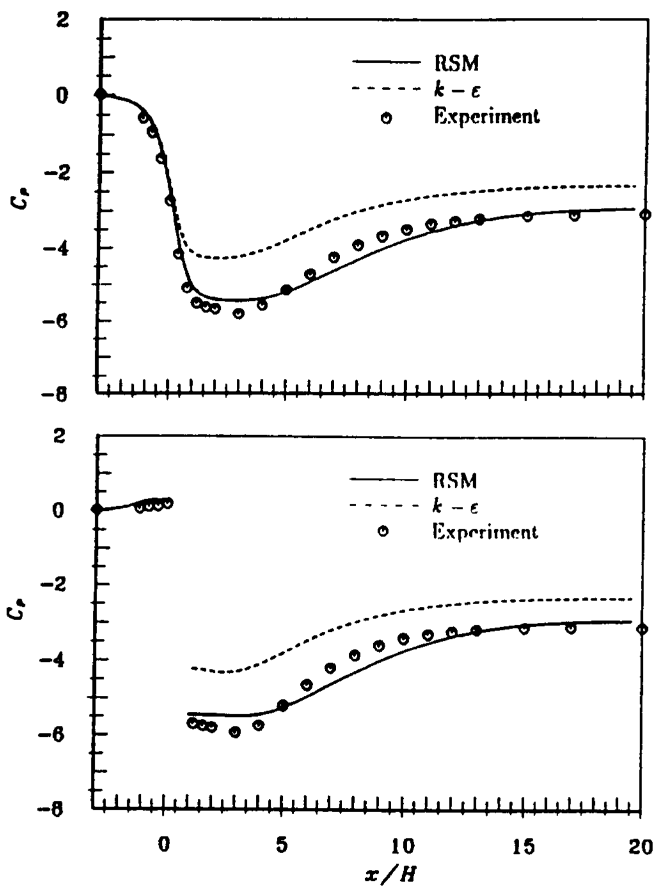

Figure 6 Pressure distribution in a flow behind a surface-mounted 2-D obstacle. ∘∘∘: Experiments (Dimaczek et al. 1989); —, --: Computations (Obi 1991)

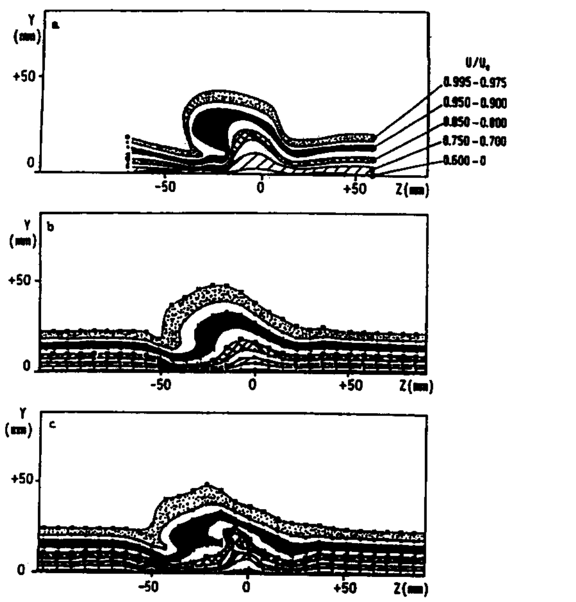

Figure 5 ISO-U contours in a turbulent boundary layer with imbedded longitudinal vortices. (a) Experiments (Bradshaw et al. 1982); (b) EVM computations; (c) DSM computations (Liandrat et al. 1985)

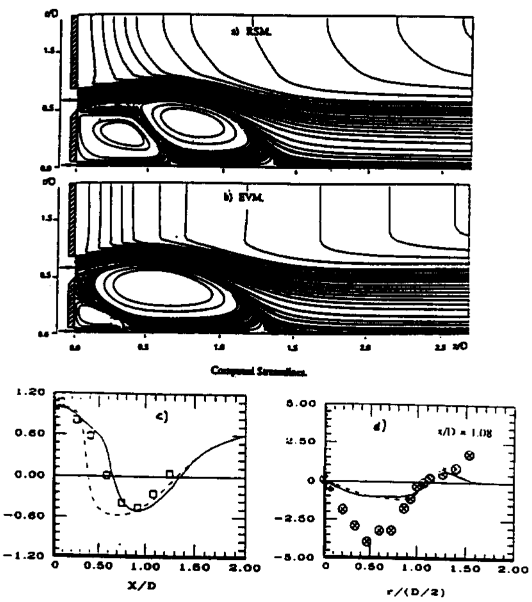

Figure 7 Computed streamlines, centerline velocity, and turbulent shear stress profile in a near-wake with a central jet. □ ⊗: Experiments; —, --: Computations (Durao et al. 1991)

A very good agreement between the measured and computed phase-averaged velocity profiles at different time instants is achieved (Figure 4a). Moreover, Figure 4b illustrates the capability of DSM to reproduce well the dynamics of the response of the turbulence stress field. Simple models like the k–ε can also reproduce a time lag of the turbulence field behind the mean flow variation (through the dynamics of k and ε), but not a fine distinction in the hysteresis of the stress anisotropy at various flow depths, as does the Re-stress model. Figure 5 compares the performances of the EVM and DSM in computing the turbulent boundary layer with imbedded longitudinal vortices (Liandrat et al. 1987). Because the stress anisotropy plays a dominant role in governing secondary motions like the longitudinal vortices, even a simple DSM shows notably more realistic predictions than EVMs.

Figure 6 presents a comparison between the computed (standard k–ε and DSM) and measured pressure distribution along the upper and bottom wall in a channel with a surface-mounted 2-D obstacle (Obi 1991). Figure 7 illustrates a notable superiority of DSM predictions, in comparison with EVMs, of the flow field and centerline velocity development in the near-wake of a disc with a central jet, though none of the models reproduced well the turbulent stress field, particularly near the stagnation region (Figure 7d) (Durao et al. 1993).

3.1.1. Some remarks on modeling wall-proximity and viscosity effects. The influences of viscosity and wall proximity upon the turbulent motion are very different by nature, yet these effects have in the past been frequently modeled jointly, because they both manifest themselves in turbulence damping. However, as is well known, if the turbulence Re number is small enough, viscosity affects all turbulent interactions, causes a departure from local isotropy (upon which some useful modeling principles rest), and promotes a consequent influence of the mean strain field upon the fine-scale turbulence. In contrast, a solid wall or a phase interface flattens the turbulent structure and imposes a selective damping, primarily of the normal-to-the-wall fluctuations, causing the turbulence to approach a 2-D state. Furthermore, the wall reflects the pressure pulsations, affecting further the stress redistribution process in the region that extends well into the turbulent flow, while the influence of viscosity upon the Re stresses at high Re-number flows remains restricted to the viscous sublayer, and even here does not seem to be very strong.

The simple two-equation models cannot differentiate these effects. Yet the demand for accurate computations of a large number of practically important flows, bounded by walls and/or phase interfaces, has prompted the appearance of many proposals for treating the near-wall region. The practice in the past 15 years has meandered between the computationally more economical wall-function bridging and a more elaborate straight-to-the-wall integration. The latter approach has relied in most cases on the introduction of a number of damping functions in terms of turbulence Reynolds number and wall distance, by means of which their effects on the various terms in the modeled transport equations are accounted for. A review by Patel et al. (1985) has shown that the two-equation models have invariably been accommodated by functions that were made to depend purely upon viscosity. Such an ill-founded practice obscures the separate effects of viscosity and wall proximity and results in poor predictions of flow regions where these effects take different relative magnitudes than in simple wall flows. A more consistent approach requires each of the effects to be modeled separately, with coefficients and functions tuned on the basis of experimental data obtained in flows where each effect can be isolated. An illustrative example is the transition from the initial to the final period of decay of isotropic turbulence, which enables a coefficient in the dissipation equation to be determined purely on the basis of the Re-number effect. Another example is the wall damping of the stress components in the near-wall region outside the viscosity-affected zone. An illustration of how to account separately for these two effects has been shown in the work of Hanjalić (1989), where the f_μ function (used commonly to damp the eddy viscosity in the sublayer region) has been decomposed into two parts, i.e., $f_\mu = f'_\mu f_w$, where f'_μ represents the effect of the low-turbulence Re number, while f_w represents the wall damping effect, assumed to be represented by the ratio of $\overline{u_2^2}/k$. The resulting expression shows a very similar behavior to the "experimental" function of Patel et al. (1985) generated from a collection of measured data.

Of course, because of high anisotropy of turbulence in the near-wall region, the Reynolds stress model (or at least the ASM), which allows computation of each stress component, has much better prospects for simulating wall proximity effects. Hanjalić and Launder (1976) differentiated to some extent the influences of two mentioned effects in their low-Re-number stress model, but departed from some of the adopted principles after they switched to a simpler three-equation model that was subsequently tested to yield satisfactory predictions of several classes of 2-D thin shear flows. Because of prohibitive computational demand, the performance of this and other full-stress, low-Re-number models were never thoroughly explored in more complex flows, for which most computors preferred to employ wall functions. More recently, the appearance of DNS data inspired several groups to re-examine and to propose new variants of the low-Re-number modifications (e.g., Launder and Shima 1989; Lai and So 1990; Launder and Tselepidakis 1991).

There has been a marked trend recently to abandon wall functions and to revert again to a more reliable integration straight to the wall even by employing locally only very simple mixing-length-type models (in spite of the well-known fact that this simple model neglects the turbulent transport by secondary motion, noticed to persist even in the very close vicinity of the walls). The PSL (Parabolic Sub-Layer) approach of Iacovides and Launder (1984a, 1984b) seemed at first to offer a possibility for employing higher-order turbulence models in the near-wall region with a substantial reduction of computation time, but was found later to lead to serious errors if applied to flows where pressure gradients normal to the flow are large (as reported, e.g., by Choi, Iacovides, and Launder 1989), in the corners of the square-sectioned curved channel).

Some fresh developments aimed at replacing the outmoded approaches of the 1970s were proposed by several groups in the last few years. By noting that the above-mentioned experimental f_μ function of Patel et al. (1985), multiplied by 0.35, follows closely the variations of $\overline{u_2^2}/k$ through the viscous sublayer, Launder et al. (1987) concluded that the shear stress damping is "largely independent of viscosity." This implies that a proper modeling of the pressure-correlation in the vicinity of a rigid wall *should* eliminate a need to modify the model for viscous damping. A way to do this is to employ the stress anisotropy invariants A_2, A_3, or "flatness" A, representing the wall-induced stress anisotropy. Some more recent proposals in this direction will be discussed in the next section.

4. Some open issues and current efforts toward the model refinements

In spite of years of intensive development and use of turbulence models, there are still many open queries that either have been resolved only partially to suit immediate needs or still pose serious challenges. Such are the problems of modeling specific effects: streamline curvature, rotation, swirl, buoyancy, compressibility, and three-dimensionality (most of these are characterized as extra-strain rates). Here no account of any of these specific effects will be given. Instead, we shall discuss some of the open issues that concern the models in general and that have major implications for the future use of turbulence models in more complex flows, with a focus on the near-wall flow regions.

4.1. Reynolds-stress transport equation

The transport equation for the Reynolds stresses can be expressed in symbolic form as

$$\frac{D\overline{u_i u_j}}{Dt} = \underbrace{\frac{\partial \overline{u_i u_j}}{\partial t}}_{L_{ij}} + \underbrace{U_k \frac{\partial \overline{u_i u_j}}{\partial x_k}}_{C_{ij}} = \underbrace{-\left(\overline{u_i u_k}\frac{\partial U_j}{\partial x_k} + \overline{u_j u_k}\frac{\partial U_i}{\partial x_k}\right)}_{P_{ij}} + \underbrace{(\overline{f_i u_j} + \overline{f_j u_i})}_{G_{ij}}$$

$$\underbrace{-2\Omega_k(\overline{u_j u_m}\epsilon_{ikm} + \overline{u_i u_m}\epsilon_{jkm})}_{R_{ij}} + \boxed{\underbrace{\overline{\frac{p}{\rho}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)}}_{\Phi_{ij}}} - \boxed{\underbrace{2\nu\overline{\frac{\partial u_i}{\partial x_k}\frac{\partial u_j}{\partial x_k}}}_{\varepsilon_{ij}}}$$

$$\underbrace{+\frac{\partial}{\partial x_k}\left[\underbrace{\nu\frac{\partial \overline{u_i u_j}}{\partial x_k}}_{D^v_{ij}} \boxed{\underbrace{-\overline{u_i u_j u_k}}_{D^t_{ij}}} \boxed{\underbrace{-\overline{\frac{p}{\rho}(u_i\delta_{jk} + u_j\delta_{ik})}}_{D^p_{ij}}}\right]}_{D_{ij}} \quad (1)$$

The terms in the boxes must be modeled, and we shall discuss briefly the current practice and some latest developments in modeling each of these terms. Much attention has been focused recently on the limiting behavior of turbulent stresses and terms in their transport equations as the wall is approached, the latter of which became fully known only after the appearance of the DNS data. The satisfaction of limiting forms is regarded nowadays as an important criterion for judging the feasibility of the models that are designed for the integration up to the wall. The limiting behavior can be evaluated by expanding the instantaneous velocity and pressure near the wall:

$$u_i = a_i + b_i y + c_i y^2 + d_i y^3 + \cdots \quad (2)$$

$$p = a_p + b_p y + c_p y^2 + d_p y^3 + \cdots \quad (3)$$

where $a_i = 0$, $b_2 = 0$ (for an incompressible fluid), but $a_p \neq 0$. Table 1 shows the wall limiting values of terms in the $\overline{u_i u_j}$ equation in the nondimensional form, which balance the budget in the near-wall region (for notations, see Equations 2 and 3). Note that $P_{ij} \sim y^3$ for all components except for $i \neq j$ if i or j is 2. Likewise, $D^t_{ij} \sim y^3$ for $i = j = 1$ or 3, D^t_{12} and D^t_{23} vary as y^4, whereas $D^t_{22} \sim y^5$. For these reasons, P_{ij} and D^t_{ij} are not included in the table.

4.1.1. Turbulent transport.

The turbulent flux term consists of the velocity transport and pressure transport

$$D^t_{ij} = D^v_{ij} + D^p_{ij} = -\frac{\partial}{\partial x_k}\overline{u_i u_j u_k} - \frac{\partial}{\partial x_k}\left[\frac{1}{\rho}(\overline{pu_i}\delta_{jk} + \overline{pu_j}\delta_{ik})\right] \quad (4)$$

Both parts of the term have to be modeled. Triple velocity moments have been measured in a variety of turbulent flows (see, e.g., Schwarz and Bradshaw 1992), whereas the pressure diffusion is still intractable to any measuring methods. Useful hints for modeling triple moments can be found in the exact transport equation derivable directly from the Navier–Stokes equation, or in general kinematic expressions for the triple-moment tensor. For this reason, most models treat both

Table 1 Near-wall behavior of pressure terms, viscous diffusion, and dissipation

ij	Φ_{ij}	Π_{ij}	D^p_{ij}	D^v_{ij}	ε_{ij}
11	$2\overline{a_p \frac{\partial b_1}{\partial x_1}} y$	$-4\overline{b_1 c_1} y$	$-\left(2\overline{a_p \frac{\partial b_1}{\partial x_1}} + 4\overline{b_1 c_1}\right) y$	$\boxed{2\overline{b_1 b_1}} + 12\overline{b_1 c_1} y$	$\boxed{2\overline{b_1 b_1}} + 8\overline{b_1 c_1} y$
22	$4\overline{a_p c_2} y$	$-4\overline{c_2 c_2} y^2$	$-(4\overline{a_p c_2} y + 4\overline{c_2 c_2} y^2)$	$12\overline{c_2 c_2} y^2$	$8\overline{c_2 c_2} y^2$
33	$2\overline{a_p \frac{\partial b_3}{\partial x_3}} y$	$-4\overline{b_3 c_3} y$	$-\left(2\overline{a_p \frac{\partial b_3}{\partial x_3}} + 4\overline{b_3 c_3}\right) y$	$\boxed{2\overline{b_3 b_3}} + 12\overline{b_3 c_3} y$	$\boxed{2\overline{b_3 b_3}} + 8\overline{b_3 c_3} y$
12	$\overline{a_p b_1}$	$-2\overline{b_1 c_2} y$	$-(\overline{a_p b_1} + 2\overline{b_1 c_2} y)$	$6\overline{b_1 c_2} y$	$4\overline{b_1 c_2} y$
23	$\overline{a_p b_3}$	$-2\overline{b_3 c_2} y$	$-(\overline{a_p b_3} + 2\overline{b_3 c_2} y)$	$6\overline{b_3 c_2} y$	$4\overline{b_3 c_2} y$
13	$\overline{a_p\left(\frac{\partial b_1}{\partial x_3} + \frac{\partial b_3}{\partial x_1}\right)} y$	$-2(\overline{b_1 c_3} + \overline{b_3 c_1}) y$	$-\overline{a_p\left(\frac{\partial b_1}{\partial x_3} + \frac{\partial b_3}{\partial x_1}\right)} y - 2(\overline{b_1 c_3} + \overline{b_3 c_1}) y$	$\boxed{2\overline{b_1 b_3}} + 6(\overline{b_1 c_3} + \overline{b_3 c_1}) y$	$\boxed{2\overline{b_1 b_3}} + 4(\overline{b_1 c_3} + \overline{b_3 c_1}) y$

Note: Terms in boxes are in balance at the wall.

terms jointly, although the arguments for designing the model are essentially those for triple moments. This does not necessarily mean that the pressure transport is negligible, although some consideration of the stress budget as well as the theoretical estimates seem to suggest this.[2] Irrespective of the analytical arguments employed for modeling, all known models use the stress gradients to express the turbulent transport terms, although some more complex models also employ the gradients of the turbulence scale or of the scale-supplying variable, such as the dissipation rate ε. The most popular is the gradient expression, attributed to Daly and Harlow (1970) (denoted hereafter by DH), by which the transport of the turbulent stress $\overline{u_i u_j}$ is expressed in term of *its* gradient:

$$\overline{u_i u_j u_k} = -C_s \frac{k}{\varepsilon} \overline{u_k u_l} \frac{\partial \overline{u_i u_j}}{\partial x_l} \tag{5}$$

with $C_s = 0.22$ (in fact, values between 0.20 and 0.25 are found in the literature). The major shortcoming of the model is the nonpreservation of the symmetry in the indices, so that the model is not rotationally invariant and the relation depends on the choice of the coordinate axes. Shir (1973) simplified further the expression by employing the isotropic transport coefficient

$$\overline{u_i u_j u_k} = -C_s \frac{k^2}{\varepsilon} \frac{\partial \overline{u_i u_j}}{\partial x_k} \tag{6}$$

(referred to also as the simple-gradient hypothesis), which, like the DH model, does not satisfy the coordinate invariance and introduces even more restrictions.

Starting with the transport equation for the triple-velocity products, which was truncated to the algebraic expression by neglecting the transport term, eliminating fourth-order cumulants on the grounds of the quasi-Gaussian assumption, and expressing the pressure–velocity correlation to be proportional to the triple moments themselves (analogous to the model of the slow pressure strain term in the stress transport equation) Hanjalić and Launder (1972) (denoted hereafter as HL) derived the invariant form of the expression for turbulent velocity transport:

$$\overline{u_i u_j u_k} = -C_s \frac{k}{\varepsilon}\left(\overline{u_i u_l} \frac{\partial \overline{u_j u_k}}{\partial x_l} + \overline{u_j u_l} \frac{\partial \overline{u_i u_k}}{\partial x_l} + \overline{u_k u_l} \frac{\partial \overline{u_i u_j}}{\partial x_l}\right) \tag{7}$$

where $C_s = 0.11$.

The HL expression was found to perform better in several types of turbulent flows (see below), but has not been extensively used for practical computations because it gives rise to a large number of component terms (in fully 3-D flow, 27 terms), particularly in non-Cartesian coordinates.

A simpler form that still satisfies the condition of coordinate-frame indifference is the expression attributed to Mellor and Herring (1973) (denoted hereafter as MH), which, unlike the HL expression, uses the isotropic transport coefficient:

$$\overline{u_i u_j u_k} = -C_s \frac{k^2}{\varepsilon}\left(\frac{\partial \overline{u_j u_k}}{\partial x_i} + \frac{\partial \overline{u_i u_k}}{\partial x_j} + \frac{\partial \overline{u_i u_j}}{\partial x_k}\right) \tag{8}$$

where $C_s \approx 0.15$. The expression requires the evaluation of only nine derivatives in a general 3-D flow.

Based on kinematic arguments and using the moment-generating function, Lumley (1978) analytically derived a more elaborate expression (strictly valid for weakly inhomogeneous flows) that does not contain adjustable constants (see also Schwarz and Bradshaw 1992):

$$\overline{u_i u_j u_k} = -\frac{1}{3C_{sL}} \frac{k}{\varepsilon}\left[G_{ijk} + \frac{C_{sL}}{4C_{sL}+5}(G_i\delta_{jk} + G_j\delta_{ik} + G_k\delta_{ij})\right] \tag{9}$$

where

$$G_{ijk} = \frac{k}{\varepsilon}\left(\overline{u_i u_l} \frac{\partial \overline{u_j u_k}}{\partial x_l} + \overline{u_j u_l} \frac{\partial \overline{u_i u_k}}{\partial x_l} + \overline{u_k u_l} \frac{\partial \overline{u_i u_j}}{\partial x_l}\right) \quad \text{and} \quad G_i = G_{ill} \tag{10}$$

in which the first term is in fact the HL expression (G_{ijk} is the same tensor as in the HL model). In the original expression of Lumley (1978), the coefficient C_{sL} is specified in the form of a function of Re_t, A_2, and A_3, whereas Schwarz and Bradshaw (1992) assign a value of 3.4 in their numerical test of the model application to a 3-D turbulent boundary layer. In comparison with other models, Lumley (1978) explicitly proposed that the pressure diffusion be expressed in the same form, i.e.,

$$\frac{1}{\rho}\overline{p u_k} = -0.4\overline{k u_k} \tag{11}$$

Magnaudet (1992) argues that none of the quoted models satisfies what he calls "asymptotic consistency"—essentially 2-D turbulence, encountered near a solid wall outside the viscous sublayer or at a free surface, when the normal velocity fluctuation vanishes and the tangential velocities remain nonzero. Magnaudet (1992) proposed a more general expression that also contains the gradients of the kinetic energy and its dissipation rate ("cross-diffusion"), which supposedly satisfies the asymptotic constraints in the near-surface region:

$$\begin{aligned} -\overline{u_i u_j u_k} = \frac{k}{\varepsilon}\Bigg[& C_{s1}\left(\overline{u_i u_l} \frac{\partial \overline{u_j u_k}}{\partial x_l} + \overline{u_j u_l} \frac{\partial \overline{u_i u_k}}{\partial x_l} + \overline{u_k u_l} \frac{\partial \overline{u_i u_j}}{\partial x_l}\right) \\ & + C_{s2}\left(\overline{u_i u_j} \frac{\partial \overline{u_k u_l}}{\partial x_l} + \overline{u_i u_k} \frac{\partial \overline{u_j u_l}}{\partial x_l} + \overline{u_j u_k} \frac{\partial \overline{u_i u_l}}{x_l}\right) \\ & + \frac{1}{k}(\overline{u_i u_j}\cdot\overline{u_k u_l} + \overline{u_i u_k}\cdot\overline{u_j u_l} + \overline{u_j u_k}\cdot\overline{u_i u_l}) \\ & \times\left(C_{s3}\frac{\partial k}{\partial x_l} + C_{s4}\frac{k}{\varepsilon}\frac{\partial \varepsilon}{\partial x_l}\right)\Bigg] \end{aligned} \tag{12}$$

Verifications of the proposed models in a variety of flows (mainly simple ones at high Re numbers) have been reported in the literature. Cormack et al. (1978) compared several models, including a complex model of their own (not discussed here) for four different flows (asymmetric channel, pipe flow, wall jet, and mixing layer). They found that their model performed best, but cautioned against its use because of its complexity. Instead, they recommended the HL model, which they found satisfactory in most cases considered. Amano and Goel (1986) compared some of the measured triple moments with those computed by the DH and HL model for the case of flow in a channel with a backward-facing step and concluded that the agreement is better in the latter case. More recently, Demuren and Sarkar computed the high-Re-number flow in a plane channel by using the DH, HL, and MH models in conjunction with several models for the pressure strain terms and found that all three models give similar results in most parts of the channel, but that the MH model reproduced best the relaxation towards isotropic conditions as the channel center is approached. Schwarz and Bradshaw (1992) measured all 10 components of the triple-velocity moments in a 3-D turbulent boundary layer on the flat floor of a duct with a 30° bend. On the basis of measured turbulent stresses (and estimated dissipation from the balance of kinetic energy), they evaluated the triple moments by the DH, HL, and Lumley expressions and found that all three models perform reasonably well in both 2-D and 3-D boundary layers and that Lumley's model performs marginally better for some components.

All the tests mentioned above were carried out for fully turbulent flows and outside the viscous wall region. Some authors share views that, excluding some specific flows and the near-wall viscosity-affected region, the turbulent transport is of minor importance, and the use of more complex models does not bring benefits in proportion to the increased consumption of computing resources (e.g., Launder (1990)). However, the DNS results of Kim et al. (1987) and Mansour et al. (1988) for a plane channel contradict the above view, at least for the two relatively low Re numbers (5600 and 14,000), indicating that the turbulent transport becomes an important part of the stress budget in the near-wall region up to $y^+ = 70$ for all components except for the spanwise one. This, of course, excludes the viscous sublayer below $y^+ < 5$ where the triple moments diminish rapidly as the wall is approached.

It is interesting to note that the DH, HL, and Magnaudet models predict a correct trend for most components, including the change of sign roughly in the buffer zone, but the intensity was severely underestimated by the HL model and much overestimated by the Magnaudet model. Figures 8 and 9 compare the performances of several models for two selected components of stress diffusion in a plane channel at Re = 5600. Contrary to the earlier-mentioned findings of Cormac et al. (1978) and Amano and Goel (1986), in the low-Re-number channel flow the DH model seems the closest to the experiments for most components, although insufficient in magnitude. An increase in the empirical coefficient could have brought the peak values into closer agreement.[3] Some improvement, both in shape and magnitude, can be achieved by using a part of the Magnaudet expression containing the gradient of ε and, to a lesser extent, of k ("cross-diffusion"), as shown in Figure 8 (cases with $C_{s2} = 0$). Of course, as indicated before, the asymptotic behavior toward the wall is not satisfied by any of the compared models. DH and HL expressions behave $\sim y^{n+4}$ as $y \to 0$, whereas the normal-to-the-wall derivatives of the triple moments approach the wall with y^{n+1}, where $n \geq 2$ is the exponent of the relevant stress component ($\overline{u_i u_j} \sim y^n$). It is interesting to note that a part of the general expression, derived by HL from the exact equation for the triple moment, contains the terms in the form $\overline{u_i u_j u_k} S_{ij}$ (where $S_{ij} = 0.5(\partial U_i/\partial x_j + \partial U_j/\partial x_i)$) that have a proper slope at the wall. These terms have been neglected in the past.

The DNS results for the plane channel (Kim et al. 1987) confirmed indeed that the pressure diffusion is negligible, as often assumed. However, because of nonzero pressure fluctuations at the wall, for $\overline{u_2^2}$ and $\overline{u_1 u_2}$ it is D_{ij}^p that closes the budget of the $\overline{u_i u_j}$ equation at the wall. The budget could be closed by an appropriate model of the $\Pi_{ij} = \Phi_{ij} + D_{ij}^p$, which will approach the wall linearly (see Table 1), though the $\overline{u_2^2}$ equation will still remain unbalanced. Because most current models use quadratic or higher-order damping of Φ_{ij}, some authors introduce a separate model of pressure diffusion

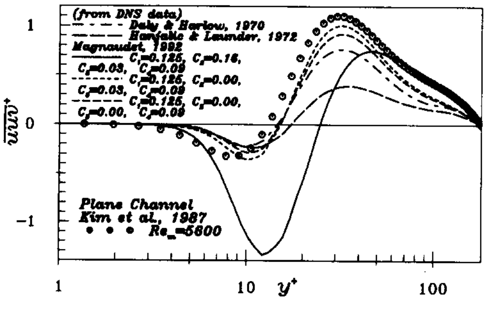

Figure 8 Triple velocity correlation $\overline{uuv}^+$ in a plane channel at Re = 5600. ○○: DNS (Kim et al. 1987); lines: modeled results

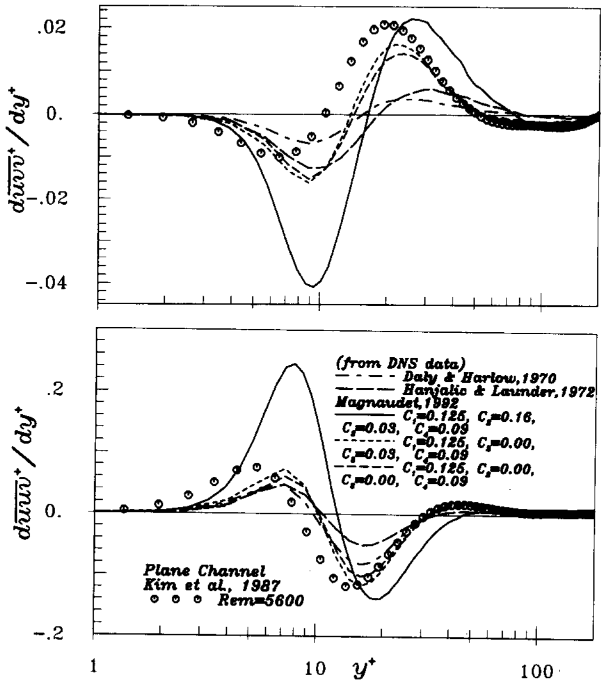

Figure 9 Turbulent velocity diffusion ($d\overline{uvv}^+/dy^+$ and $d\overline{uuv}^+/dy^+$) in a plane channel

in a discriminative form (by means of unit vectors normal to the wall), which makes contributions only to the $\overline{u_2^2}$ and $\overline{u_1 u_2}$ components (Launder and Tselepidakis 1987; Lai and So 1990).

A promising approach is that proposed by Nagano and Tagawa (1991) based on the structural characteristics of the shear-generated turbulence, which allows the evaluation of all mixed components of triple-velocity products $\overline{u_i u_j u_k}$ from the skewness factors of velocity fluctuations. Tested in several types of flows, the method showed good agreement with measurements. The method requires, however, the modeling and solution of the transport equation for the third moment of each velocity fluctuation $\overline{u_i^3}$. Although this is a much simpler task than solving the transport equations for all triple correlations, the equation set becomes too cumbersome for application in more complex flows, not to mention still unresolved uncertainties in modeling equations for the triple moments.

4.1.2. Pressure strain.

The modeling of the pressure-straining redistribution, defined as

$$\Phi_{ij} = \Phi_{ij,1} + \Phi_{ij,2} + \Phi_{ij,w} = \overline{\frac{p}{\rho}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)} \tag{13}$$

remains the most uncomfortable task. The term is usually decomposed into three parts, which follow from the exact Poisson equation for the pressure fluctuations. The first two terms are strictly the volume integrals of the two-point correlations, whereas the third term represents the surface integral and is effective only in the vicinity of a solid wall or interphase surface. Various proposals for modeling each term were put foward over the years, but most models differ only in minor, although not unimportant, details and in the degree to which they satisfy realizability, coordinate-frame indifference, and other constraints. Invariably, all proposals are based on intuitive arguments formulated in a general way by imposing the kinematic constraints from tensor algebra: symmetry, zero trace, Caley–Hamilton theorem, etc. Proposals originating from the groups of Launder and Lumley lead the field, with

Table 2 Coefficients in models for Φ_{ij}

	$\Phi_{ij,1}$		$\Phi_{ij,2}$						
	Linear	Quadr.	Linear				Quadratic		Cubic
Author(s)	C_1	C_1'	C_2	C_3	C_4	C_5	C_6	C_7	C_8
LRR (IP)	1.8	—	—	0.8	0.6	0.6	—	—	—
LRR (QI)	1.8	—	—	0.8	0.873	0.655	—	—	—
SSG	1.7	−1.05	0.9	$0.8 - 0.625A_2^{1/2}$	0.625	0.2	—	—	—
ChL, SL	0.5β	0.25γ	—	0.8	$0.6(1 + 0.8A^{1/2})$	—	0.2	0.2	—
CL	$3.1(A_2A)^{1/2}$	1.2	—	0.8	0.6	0.866	0.2	0.2	1.2
LT	$6.3AF^{1/2}(1 - f)$	$0.7C_1$	—	0.8	0.6	0.866	0.2	0.2	$2r$

Note: $A = 1 - 9/8(A_2 - A_3)$; $A_2 = a_{ij}a_{ji}$; $A_3 = a_{ij}a_{jk}a_{ki}$; $f = \max(1 - \mathrm{Re}_t/140, 0)$; $F = \min(0.6, A_2)$; $r = \min(0.6, A)$; $\beta = \beta(A_2, A_3, A, \mathrm{Re}_t)$; $\gamma = \gamma(A_2, A_3, A)$ (see Choi and Lumley, 1984).
Abbreviations: LRR: Launder, Reece, and Rodi 1975; SSG: Speziale, Sarkar, and Gatski, 1991; ChL: Choi and Lumley, 1984; SL: Shih and Lumley, 1985; CL: Craft and Launder, 1991; LT: Launder and Tselepidakis, 1991.

some notable recent contribution from the NASA groups (e.g., Shih et al. (1992); Shih and Lumley 1992; 1993). An attempt to give a comparative tabular review of some of the most popular and recent models is given in Table 2. Here, we shall briefly discuss major points and some of the open issues relevant to the complex flows in engineering and aeronautics.

The first part of the volume integral (the "slow" part) $\Phi_{ij,1}$ contains only velocity fluctuations and causes turbulence to approach an isotropic state itself, irrespective of turbulence generation. For that reason, $\Phi_{ij,1}$ is usually modeled in terms of the stress anisotropy tensor $a_{ij} = \overline{u_iu_j}/k - 2/3\delta_{ij}$ and its first and second invariants $A_2 = a_{ij}a_{ji}$ and $A_3 = a_{ij}a_{jk}a_{ki}$. A general nonlinear model follows from the Cayley–Hamilton theorem (Lumley 1978) and can be expressed as

$$\Phi_{ij,1} = -\varepsilon[C_1a_{ij} + C_1'(a_{ij}a_{jk} - \tfrac{1}{3}A_2\delta_{ij})] \tag{14}$$

Most earlier models employ only the first term, which is essentially Rotta's (1951) linear model with the values of C_1 between 1.5 and 3.5. This form does not satisfy the 2-D turbulence limit (that $\Phi_{qq,1}$ should equal zero if u_{qq} is zero, with α denoting the principal stress axis). A way to satisfy this requirement is to consider C_1 as a function $C_1 = Af(A_2, A_3, \mathrm{Re}_t)$ (Launder 1990) where $A = 1 - 9/8\,(A_2 - A_3)$ vanishes in the 2-D limit. Alternatively, this can be achieved by replacing ε in Equation 14 by $\tilde{\varepsilon} = \varepsilon - 2\nu\ (\partial k^{1/2}/\partial x_n)^2$. The two-component limit can be directly satisfied by the nonlinear model if expressed in a specific form (Reynolds 1984).

Lumley (1978) showed analytically that C_1 cannot be a constant and proposed an expression for C_1 in terms of A_2, A_3, and Re_t for the linear model. Weinstock (1982) calculated from first principles that C_1 differs even for different components—which was also confirmed by DNS results for several wall flows (Kim et al. 1987; Spalart 1988)—and suggested that $\Phi_{ij,1}$ should be modeled in terms of the anisotropy of the dissipation rate tensor $e_{ij} = \varepsilon_{ij}/\varepsilon - 2/3\delta_{ij}$, instead of a_{ij}. The same proposal came from Lee and Reynolds (1985), whose DNS of homogeneous turbulence, subjected to irrotational strain and subsequently relaxed, showed that $\Phi_{ij,1}$ is well correlated with $e_{ij}\sqrt{\mathrm{Re}_t}$. Because the considered flow had a very low Re number, it is questionable how the model would perform at higher Re numbers and in inhomogeneous flows.

The inclusion of the nonlinear term brings in more flexibility but also an additional coefficient C_1' and new uncertainties. Speziale, Sarkar, and Gatski (1991) (denoted as SSG) found that the best predictions of several homogeneous flows are achieved with C_1' having an opposite sign from C_1 ($C_1 = 1.7$, $C_1' = -1.05$) (found also by Hanjalić et al. 1992). Launder and Tselepidakis (1991) (also Fu et al. 1987; Craft and Launder 1991) used $C_1' = 0.7 \cdot C_1$ where $C_1 = C_1(A_2, A_3, \mathrm{Re}_t)$ (see later). More recently, Demuren and Sarkar (1993) reported that the SSG model, used with wall functions, reproduced well the experimental data of Laufer for a plane channel at Re = 52,000 (outside the viscosity-affected region).

The rapid part of the pressure-strain redistribution $\Phi_{ij,2}$ is usually represented in the form

$$\Phi_{ij,2} = \frac{\partial U_l}{\partial x_m}(a_{lj}^{mi} + a_{li}^{mj}) \tag{15}$$

which presumes that the mean velocity is nearly homogeneous. Fourth-rank tensors are expressed in terms of the turbulent stress anisotropy a_{ij} in a linear or higher-order form, and coefficients are sought from various kinematic and symmetry constraints. Practically, all proposed expressions can be written in the general form containing the mean rate of strain S_{ij}, mean vorticity Ω_{ij} and stress anisotropy tensors a_{ij}:

$$\begin{aligned}\Phi_{ij,2} = {} & C_2Pa_{ij} + C_3kS_{ij} \\ & + C_4k(a_{ik}S_{jk} + a_{jk}S_{ik} - \tfrac{2}{3}a_{kl}S_{kl}\delta_{ij}) \\ & + C_5k(a_{ik}\Omega_{jk} + a_{jk}\Omega_{ik}) \\ & + C_6k(a_{ik}a_{kl}S_{jl} + a_{jk}a_{kl}S_{il} - 2a_{kj}a_{li}S_{kl} - 3a_{ij}a_{kl}S_{kl}) \\ & + C_7k(a_{ik}a_{kl}\Omega_{jl} + a_{jk}a_{kl}\Omega_{il}) \\ & + C_8k[a_{nn}^2(a_{ik}\Omega_{jk} + a_{jk}\Omega_{ik}) + \tfrac{3}{2}a_{mi}a_{nj}(a_{mk}\Omega_{nk} + a_{nk}\Omega_{mk})]\end{aligned} \tag{16}$$

where

$$P = \overline{u_iu_j}\frac{\partial U_i}{\partial x_j}, \qquad S_{ij} = \frac{1}{2}\left(\frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i}\right),$$

$$\Omega_{ij} = \frac{1}{2}\left(\frac{\partial U_i}{\partial x_j} - \frac{\partial U_j}{\partial x_i}\right) \tag{17}$$

The expression reduces to the simplest "isotropization of production" (IP) model of Launder, Reece, and Rodi (1975) (denoted as LRR), when $C_2 = 0$, $C_3 = 0.8$, $C_4 = C_5 = 0.6$ and C_6, C_7, C_8 are zero, or to their "quasi-isotropic" (QI) model when $C_2 = 0$, $C_3 = 0.8$, $C_4 = 0.872$, $C_5 = 0.6545$, and C_6, C_7, $C_8 = 0$. Both models have been extensively in use in spite of noted deficiencies (i.e., the need to account for wall reflection, poor performances in swirling flows in impingement regions (e.g., Craft and Launder 1991). Shih and Lumley (1985) argued that the linear expression cannot satisfy the realizability conditions as the turbulence approaches a two-component state, and proposed a more general form that contains quadratic terms (see Table 2).

Speziale, Sarkar, Gatski (1991) proposed a model containing the same linear terms as the LRR model, but with some coefficients dependent on the turbulent stress invariants and turbulence production (hence, "quasi-linear"). They found that the model satisfies both the homogeneous and wall-equilibrium flows without a need to introduce the wall-reflection correction as used by LRR. Like the LRR model, the SSG model does not satisfy the realizability constraints. Of the nonlinear models, we will mention several variants of the Shih and Lumley models and also of the Launders group model. The latest version of the Shih and Lumley model (1992) strictly satisfies both the Schwartz inequality and the two-component realizability constraints. The same claim is made for the higher-order model of Fu, Launder, and Tselepidakis (1987) (also Craft and Launder 1991; Launder and Tselepidakis 1991), which is essentially cubic in the turbulent stress, and which contains only one freely determinable coefficient. This model was developed with the specific aim of accommodating more complex 3-D turbulent flows. In addition to better satisfying the kinematic constraints, both nonlinear models are claimed to eliminate some deficiencies of the linear models (see, e.g., Shih and Lumley 1992; Launder 1990), but are much more complex for use in engineering computation.

The wall-reflection term $\Phi_{ij,w}$ (introduced by LRR, Shir (1973), and Gibson and Launder (1978) to compensate for notable differences in stress anisotropy in the wall and far-from-the-wall homogeneous flows at comparable shear rates) was also supposed to simulate the surface integral of the Poisson equation $\Phi_{ij,w}$. The net effect is a selective damping of the fluctuations only in the direction normal to the wall. In order to introduce the damping effect with a selective orientation, all models contain a function that is related to the normal distance from the wall and is not coordinate-frame invariant. The addition of $\Phi_{ij,w}$ has been generally regarded as an unavoidable necessity, but because of its relation to the local wall distance, it represents a major weakness. This deficiency, as well as more recent experimental evidence that the wall damping effect is smaller than considered hitherto (e.g., Speziale, Sarkar, and Gatski (1991)), prompted a new tendency

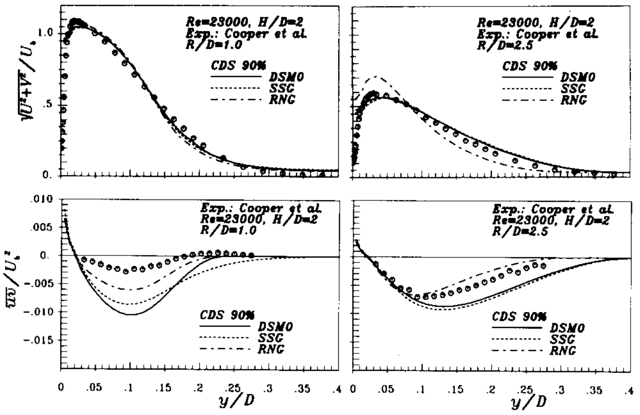

Figure 11 Mean velocity and shear stress in an impinging jet—computations with Re-stress models (Jakirlić and Hanjalić 1993). DSMO: basic model; SSG: Φ_{ij} model of Speziale, Sarkar and Gatski 1991; RNG: RNG-modifications of ε-equation and diffusion coefficients

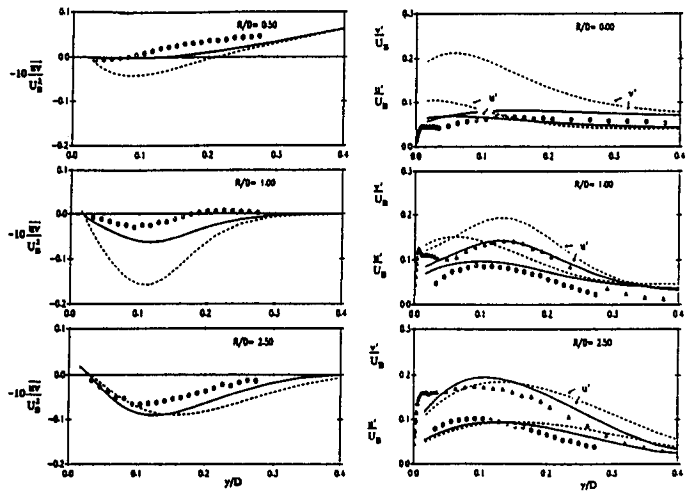

Figure 10 The shear stress and rms of u_1 and u_2 in an impinging jet (Craft and Launder 1992). △, ∘ u_1, u_2: experiments. Computations: - - -: basic model; ——: new reflection model

to eliminate this term by constructing better forms of the models of $\Phi_{ij,1}$ and $\Phi_{ij,2}$, which should account for the wall reflection in a general and invariant manner. Some success in this direction has been claimed for simple wall-parallel flows, but it seems that predictions of turbulence properties in a stagnation region require even a more elaborate model of $\Phi_{ij,w}$, as illustrated below.

The initial model of $\Phi_{ij,w}$ was introduced to accommodate the wall-reflection effect in wall-parallel flows (but outside the viscosity-affected region) and was designed to diminish the transfer of energy into or out of the normal-to-the-wall component, irrespective of how the turbulence energy is produced and into which component it is being fed. Craft and Launder (1991) argued that in a stagnation region, it is the normal component that receives most of the energy and then shares it with other components by the redistributive action of the fluctuating pressure. Hence, the transfer out of the normal component is most intensive, whereas the original $\Phi_{ij,w}$ acts to hinder this transfer. Craft and Launder (1991) proposed a more general form of $\Phi_{ij,w}$, which greatly improved the predictions of the stress components and, consequently, of heat transfer in the stagnation region of a jet impinging normally on a flat surface, as illustrated in Figure 10. The model, however, still contains a function in terms of the local wall distance. It should be pointed out that the SSG model cannot remedy this deficiency, as shown in Figure 11.

A way to construct an invariant form of the $\Phi_{ij,w}$ may be to employ the "flatness" of the stress-anisotropy A instead of the local wall distance. However, because the pressure-strain interaction is of elliptic nature and is governed by the integral of the Poisson equation over the whole visible flow domain, the real effects of the wall can hardly be accounted for by models involving only the local flow properties. The model of the surface integral in the regions close to a solid boundary in terms of local properties, even if they involve the local distance from the wall, is even more questionable due to strong inhomogeneities. Durbin (1991, 1993) proposed a way to account for the nonlocal nature of the effect and for the near-wall inhomogeneity by solving—in addition to the modeled set of transport equations—an additional second-order differential equation ("elliptic relaxation equation"), for the function $f_{ij} = \Phi_{ij}/k$

$$L^2\nabla^2 f_{ij} - f_{ij} = \frac{1}{k}\,\Phi_{ij}^{\mathrm{LRR}} \tag{18}$$

where for the source term Φ_{ij}^{LRR} the standard (local) model of LRR for the Φ_{ij} was applied (other models can also be used), and L is the turbulence length scale. Application of the model with a new definition of turbulence scales in the viscous wall region (see below) for several wall thin shear flows produced good agreement with the DNS results and experiments.

Modifications of the pressure-strain term to account for low-Re-number effect and to satisfy the two-component limit at the wall is still a subject of controversy. Essentially, viscosity does not appear in the Poisson equation for the fluctuating pressure, and consequently one may argue that viscosity does not *directly* affect the pressure-straining process. However, the wall damping within the viscous sublayer differs in intensity, and, probably, in mechanism from that in the outer turbulent wall zone. Let us confine our attention to the behavior of exact terms in the $\overline{u_i u_j}$ equation, particularly of terms involving the fluctuating pressure in the limit as the wall is approached. As seen in Table 1, at the wall, $\Phi_{ij} = 0$ for all $i = j$, but $\Phi_{12} \neq 0$ and is in balance with the pressure diffusion. Contrary to that, as shown in Table 3 (for notation, see Equations 2 and 3), $a_{ij} \neq 0$ for all $i = j$, but $a_{12} = 0$. Obviously, Rotta's linear model of $\Phi_{ij,1}$ with $C_1 = \text{const.}$ (and $C_1' = 0$; see Equation 14) cannot satisfy the wall limiting behavior of the exact pressure-strain term. It is also obvious that it will be too demanding to define a selective function C_1 that will distinguish the diagonal from the off-diagonal components of Φ_{ij}. As mentioned earlier, Launder and Tselepidakis (1991) enforce C_1 to satisfy the wall limit by multiplying it by A, but also by a Re_t function, $C_1 = 6.3AF(1 - f)$, where $F = \min(0.6, A_2^{1/2})$ and $f = \max(1 - Rt/140, 0)$. In conjunction with their nonlinear model for $\Phi_{ij,2}$, and modifications of the ε equation, as discussed later, they reproduce very well the DNS data of Kim et al. (1987) for Re = 5600 and Re = 14,000.

As seen in Table 1, the sum of pressure-strain and pressure diffusion, $\Pi_{ij} = \Phi_{ij} + D_{ij}^p$, goes to zero (though not at the same rate) for all i and j. This is why some researchers have proposed to model Π_{ij} instead of Φ_{ij}, or at least to interpret the model as simulating Π_{ij} and not Φ_{ij}. Lumley and Newman (1977) modeled jointly Π_{ij}–ε_{ij}, whereas Launder and Shima (1989) proposed a model of Φ_{ij}–ε_{ij}. The latter model employs the standard high-Re-number LRR model of Φ_{ij} and the Gibson and Launder (1978) model of $\Phi_{ij,w}$, but with all coefficients dependent on A_2 and A_3.

4.2. Dissipation rate

The modeling of the dissipation rate in the Re-stress equation and the definition of the characteristic turbulence scale(s) used to model various terms in transport equations depend on the choice of the scale-supplying variable. Among a variety of variables, the most widely used is the dissipation rate of turbulent kinetic energy ε, governed by the exact equation

$$\underbrace{\frac{D\varepsilon}{Dt} = \frac{\partial \varepsilon}{\partial t}}_{L_\varepsilon} + \underbrace{U_k \frac{\partial \varepsilon}{\partial x_k}}_{C_\varepsilon} = \boxed{\underbrace{-2\nu\left(\overline{\frac{\partial u_i}{\partial x_l}\frac{\partial u_k}{\partial x_l}} + \overline{\frac{\partial u_l}{\partial x_i}\frac{\partial u_l}{\partial x_k}}\right)\frac{\partial U_i}{\partial x_k}}_{P_{\varepsilon 1} + P_{\varepsilon 2}}}$$
$$\boxed{\underbrace{-2\nu \overline{u_k \frac{\partial u_i}{\partial x_l}} \frac{\partial^2 U_i}{\partial x_k \partial x_l}}_{P_{\varepsilon 3}}} \boxed{\underbrace{-2\nu \overline{\frac{\partial u_i}{\partial x_k}\frac{\partial u_i}{\partial x_l}\frac{\partial u_k}{\partial x_l}}}_{P_{\varepsilon 4}}} - \boxed{\underbrace{2\overline{\left(\nu \frac{\partial^2 u_i}{\partial x_k \partial x_l}\right)^2}}_{Y}}$$
$$\underbrace{+\frac{\partial}{\partial x_k}\left(\underbrace{\nu \frac{\partial \varepsilon}{\partial x_k}}_{D_\varepsilon^\nu} \boxed{\underbrace{-\overline{u_k \varepsilon}}_{D_\varepsilon^t}} \boxed{\underbrace{-\frac{2\nu}{\rho}\overline{\frac{\partial p}{\partial x_i}\frac{\partial u_k}{\partial x_i}}}_{D_\varepsilon^p}}\right)}_{D_\varepsilon} \qquad (19)$$

where the terms in the boxes are to be modeled. The first attempts to model the ε-equation tended to draw the information from its exact form, but the outcome has in the past relied mainly on intuition. One of the reasons was the lack of information about complex interactions represented by various terms in Equation 20, but another was a limited number of variables available in the single-point closure technique to model unknown terms. Besides, what we need is the scale of the energy-containing eddies and the equation for the energy transfer from these eddies down the spectrum, which coincides with the dissipation only under the conditions of spectral equilibrium (Hanjalić, Launder, and Schiestel 1980). It is remarkable that the modeled ε-equation has persistently remained in use in its very primitive form as it first appeared for high Re-number flows (Davidov 1961; Hanjalić and

Table 3 Behaviour of stress anisotropy a_{ij} at the wall and its vicinity

ij	Wall value	a_{ij}	Higher-order term
11	$\frac{2}{3}\frac{2\overline{b_1 b_1} - \overline{b_3 b_3}}{\overline{b_1 b_1} + \overline{b_3 b_3}}$	$+$	$\frac{4(\overline{b_3 b_3}\,\overline{b_1 c_1} - \overline{b_1 b_1}\,\overline{b_3 c_3})}{(\overline{b_1 b_1} + \overline{b_3 b_3})^2} y$
22	$-\frac{2}{3}$	$+$	$\frac{2\overline{c_2 c_2}}{\overline{b_1 b_1} + \overline{b_3 b_3}} y^2$
33	$\frac{2}{3}\frac{2\overline{b_3 b_3} - \overline{b_1 b_1}}{\overline{b_1 b_1} + \overline{b_3 b_3}}$	$-$	$\frac{4(\overline{b_3 b_3}\,\overline{b_1 c_1} - \overline{b_1 b_1}\,\overline{b_3 c_3})}{(\overline{b_1 b_1} + \overline{b_3 b_3})^2} y$
12	0	$+$	$\frac{2\overline{b_1 c_2}}{\overline{b_1 b_1} + \overline{b_3 b_3}} y$
23	0	$+$	$\frac{2\overline{b_3 c_2}}{\overline{b_1 b_1} + \overline{b_3 b_3}} y$
13	$\frac{2\overline{b_1 b_3}}{\overline{b_1 b_1} + \overline{b_3 b_3}}$	$+$	$\frac{2(\overline{b_1 b_1} + \overline{b_3 b_3})(\overline{b_1 c_3} + \overline{b_3 c_1}) - 4\overline{b_1 b_3}(\overline{b_1 c_1} + \overline{b_3 c_3})}{(\overline{b_1 b_1} + \overline{b_3 b_3})^2} y$

Launder 1972):

$$\frac{D\varepsilon}{Dt} = (C_{\varepsilon 1}P - C_{\varepsilon 2}\varepsilon)\frac{\varepsilon}{k} + \frac{\partial}{\partial x_k}\left(C_\varepsilon \frac{k}{\varepsilon}\overline{u_k u_l}\frac{\partial \varepsilon}{\partial x_l}\right) \tag{20}$$

Although regarded from the very beginning as too simple for representing the dynamics of the turbulence energy decay rate and for serving as a source of the characteristic length scale, the modeled ε-equation exhibited a surprising degree of generality. Numerous tests have revealed also its limitations and shortcomings, urging a search for a more general form. So far, however, none of the proposals for its refinements and generalizations has been able to withstand the scrutiny.

The first term in Equation 20 is supposed to model the difference between the production of ε due to vortex self-stretching $P_{\varepsilon 4}$ and viscous destruction Y. These two terms dominate the dynamics of ε at high Re numbers and balance the transport terms so that success of the model depends on a plausible modeling of the difference of two large quantities. For that reason, instead of modeling individually each term in the exact Equation 20, the efforts in the past have concentrated on modeling the *net* effect of major terms. Nevertheless, further improvement of the model of the ε equation, particularly for low-Re numbers, should benefit from term-by-term analysis. The major source of ε, known to originate from the promotion of energy transfer through the spectrum due to the self-stretching of vortex filaments, $P_{\varepsilon 4}$, has been modeled in terms of turbulence energy production P (scaled by the turbulence time scale $\tau = k/\varepsilon$). This is a radical over-simplification that imposes a number of constraints. Firstly, it presumes that the process is equally affected by both the linear and angular mean-flow deformation. Secondly, it does not provide room for modeling separately the effect of the mean-flow vorticity. Furthermore, it implies that the same time scale controls the process of production and viscous destruction of ε. A more general model is needed that would remove the mentioned and other limitations as well as allow the effect of turbulence anisotropy and other parameters to play adequate roles. In fact, a model for the production of ε in terms of turbulence (rather than mean-flow) parameters should be regarded as physically most justified and one that would satisfy the principle of physical coherence. As is well known, Lumley and Khajeh Nouri (1974) proposed to model the source of ε solely in terms of the turbulence anisotropy second invariant $A_2 = a_{ij}a_{ij}$. Although well argued, the idea did not prove to be very helpful when tested. However, like some other ideas that rest on sound physical reasoning, it deserves to be reincarnated and retested. A possible general source term can be recast in a functional form (for high Re-numbers):

$$P_\varepsilon = f\left[\overline{u_i u_j}\frac{\partial U_i}{\partial x_j}\frac{\varepsilon}{k},\ k\left(\frac{\partial U_i}{\partial x_j}\right)^2,\quad k\left(\epsilon_{ijk}\frac{\partial U_i}{\partial x_j}\right)\left(\epsilon_{mnk}\frac{\partial U_m}{\partial x_n}\right),\right.$$
$$\left. S\varepsilon,\ \Omega\varepsilon,\ A_2\frac{\varepsilon^2}{k},\ldots,\right] \tag{21}$$

where $S = (S_{ij}S_{ji})^{1/2}$ and $\Omega = (\Omega_{ij}\Omega_{ji})^{1/2}$ are the moduli of the mean rate of strain S_{ij} and of mean vorticity Ω_{ij}, respectively. In comparison with the standard form of P_ε, the introduction of new terms offers more flexibility in modeling various effects, and brings into play the time scale of the mean-flow deformation $\tau_m = (\partial U_i/\partial x_j)^{-1}$ or S^{-1}, in addition to the turbulence time scale $\tau = k/\varepsilon$.

Several previous proposals to introduce new terms in the forms listed in Equation 21 or others yielded improvements in predictions of some classes of flows (including 2-D separated and 3-D curved channel flows), but were eventually found to worsen others. Pope (1978) proposed a mean vortex stretching term $C_{\varepsilon 4}(k^2/\varepsilon)S_{ij}\Omega_{jk}\Omega_{ki}$ ($C_{\varepsilon 4} = 0.79$) to resolve the "round jet anomaly" (smaller rate of spread as compared with a plane jet). Hanjalić and Launder (1980) argued that a term $C_{\varepsilon 4}(\epsilon_{ijk}\partial U_i/\partial x_j)(\epsilon_{mnk}\partial U_m/\partial x_n)$ augments the generation of ε by irrotational strain as compared with the shear strain, accounting thus far also for the streamline curvature. The introduction of this term in the dissipation equation produced improvements in several classes of evolving and nonequilibrium boundary-layer flows, reduced the excessive rate of spread of round jet, and increased the reattachment length in a flow behind a back step. Recent tests of this term in the computation of separated boundary layers helped to produce a separation (though with a substantial increase in $C_{\varepsilon 4}$), "but left serious errors in the separation region" (Atkinson and Castro 1991). Neither of the two mentioned proposals withstands the tests in the decay of isotropic turbulence in a rotating frame and in homogeneous shear flow in a rotating frame (Speziale, Rishi, and Gatski 1990). A proposal from Bardina (1988) to modify both coefficients in the ε equation in terms of the intensity of mean vorticity, i.e., $C_{\varepsilon 1} = 1.50 - 0.015(k/\varepsilon)\Omega$ and $C_{\varepsilon 2} = 1.83 - 0.15(k/\varepsilon)\Omega$ (where $\Omega = \sqrt{\frac{1}{2}\Omega_{ij}\Omega_{ji}}$), seems to account better for the rotational strain. Earlier experiments at UMIST (Fu, Launder, and Tselepidakis 1987), with the inclusion of an additional source term in the form $0.6A_2\varepsilon/k$ and a decrease in the value of $C_{\varepsilon 1}$ from the standard 1.4 to 1.1, have also resulted in the diminishing of anomalous predictions of the axisymmetric jet. A more recent version, apparently recommended at present for free flows, replaces $C_{\varepsilon 2}$ by the function $1.92/(1 + 0.65AA_2^{1/2})$ with $C_{\varepsilon 1} = 1$, or $C_{\varepsilon 1} = 0.35$, but with an additional production term in the form $0.35\nu_t(\varepsilon/k)(\partial U_i/\partial x_j)^2$ (essentially the second term in Equation 21; see Craft and Launder 1991).

Another possibility lies in the revision of the model of the diffusion term. Neglecting transport terms in the exact equation for the correlation $Q_k = \nu u_k(\partial u_i/\partial x_j)^2$ yields the expression

$$Q_k = -C_\varepsilon \frac{k}{\varepsilon}\left(\overline{u_k u_j}\frac{\partial \varepsilon}{\partial x_j} + C_q \varepsilon\frac{\partial \overline{u_k u_j}}{\partial x_j} + Q_j\frac{\partial U_k}{\partial x_j}\right). \tag{22}$$

Only the first term has been retained in the standard model, a simplification justified for wall boundary layers on the basis of an order-of-magnitude estimate; but for more complex flows, these arguments do not hold. No serious testing of the above, more general formulation has been reported, although some numerical experiments with other additional diffusion-type terms, formulated in an ad hoc manner, e.g.,

$$\frac{\varepsilon}{k}\frac{\partial}{\partial x_j}\left(\frac{\mu_t}{\rho}\frac{\partial k}{\partial x_j}\right);\qquad \frac{\partial}{\partial x_j}\left(k\frac{\partial k}{\partial x_j}\right);\qquad \left(\frac{\partial k}{\partial x_j}\right)^2 \tag{23}$$

to improve the predictions of nonequilibrium wall boundary layers did not yield the desired effects. Some attempts in a different direction—to ensure the experimentally detected constancy of the length-scale gradient in the near-wall region in strongly nonequilibrium flows and close to the reattachment points in separating flows—through the dependence of C_ε on the ratio P/ε produced only modest improvements. A recent introduction of an extra term (Launder 1990) in the ε-equation, tailored to compensate excessive departure of the length-scale gradient in the near-wall region, improved predictions of flow and heat transfer in some separating wall flows, but its dependence on the near-wall distance makes it inapplicable in more complex geometries.

New chances lie in revisiting some other discarded or not fully explored ideas of the past. Following the direct-interaction

(DI) approximation of Kraichnan, Yoshizawa (1988) derived arguments for mixed- or cross-diffusion terms in both the k- and ε-equation. Another route has been followed by Aupoix et al. (1987) who derived the ε-equation for homogeneous turbulence following the MIS (Methode Integrale Spectrale) approach, which assumes the evolution of an a priori defined energy spectrum shape.

The Renormalization Group Theory (RNG) of Yakhot and Orszag (1986) and Yakhot et al. (1992) seems to offer a new theoretical support to the basic form of the ε-equation, and also perspectives to better account for the effects of extra strain rates. Without going deeper into the theoretical arguments and mathematical derivation, which is beyond the scope of the present review, it is worthwhile to look at the implications of the theory on the final form of the modeled set of equations. The first conclusion put foward by the authors was that the theory yielded the same form of the dissipation equation for high Re numbers as Equation 20 with an additional term, and produced the numerical values of the coefficients—considerably different from the standard values—without employing any experimental results.[4] However, although seemingly very precise (given to four decimal digits, e.g., $C_{\varepsilon 1} = 1.063$, $C_{\varepsilon 2} = 1.7215$!), the coefficients had to be brought closer to the conventional values to reproduce some simple flows, and new values have recently been proposed, namely, $C_{\varepsilon 1} = 1.4$ and $C_{\varepsilon 2} = 1.68$ (Orszag et al. 1993). Besides, Speziale (1991) commented that the original value of $C_{\varepsilon 1}$ was too close to 1, which defines the singularity of the ε-equation. A considerably smaller value of $C_{\varepsilon 2}$ from the standard value of about 1.9 is compensated for by a higher diffusion coefficient (by 80%) and the additional term. However, this model produces the exponent of the isotropic decay $n = 1.47$, which is far higher than the experimentally obtained value of 1.1–1.25. The major implication of the RNG theory on the ε equation is, however, a new term

$$R = \frac{\eta(1 - \eta/\eta_0)}{1 + \beta\eta^3} \frac{P\varepsilon}{k} \tag{24}$$

where $\eta = Sk/\varepsilon$ is the ratio of turbulence to mean strain time scale and $\beta \approx 0.015$ is the constant. η_0 was chosen as 4.38 and represents a typical value in homogeneous shear flows. This additional term deserves attention because it changes sign depending on whether the time-scale ratio η is greater or smaller than the homogeneous value η_0, distinguishing in such a way the small from the large strain rates. This feature has probably contributed more than other modifications to an apparent success of the model to predict the appropriate length of recirculating zones of several separating flows, as compared with standard models (see, e.g., Orszag et al. 1993). However, the RNG-derived k–ε model (both the high- and low-Re number versions) brought only marginal improvement of the velocity field in a flow through a staggered tube bank, as shown in Figure 12 (but no improvement in the badly predicted shear stress field). Incorporated in the standard high-Re-number DSM, the RNG ε-equation produced inconclusive effects in the impinging jet (see Figure 11). This and other experience suggest that more research along these lines may result in a more general and universal form of the dissipation equation.

Low-Re-number modifications of the dissipation equation, first proposed by Jones and Launder (1972) and subsequently modified by Launder and Sharma (1974), seems still to be the most frequently used in conjunction with the *two-equation* models. By adopting a zero value of ε at the wall, Jones and Launder (1972) essentially do not solve the equation for true dissipation ε, but for its homogeneous part $\tilde{\varepsilon} = \varepsilon - 2\nu(\partial k^{1/2}/\partial x_n)^2$. Many other forms of low-Re-number ε-equation have been

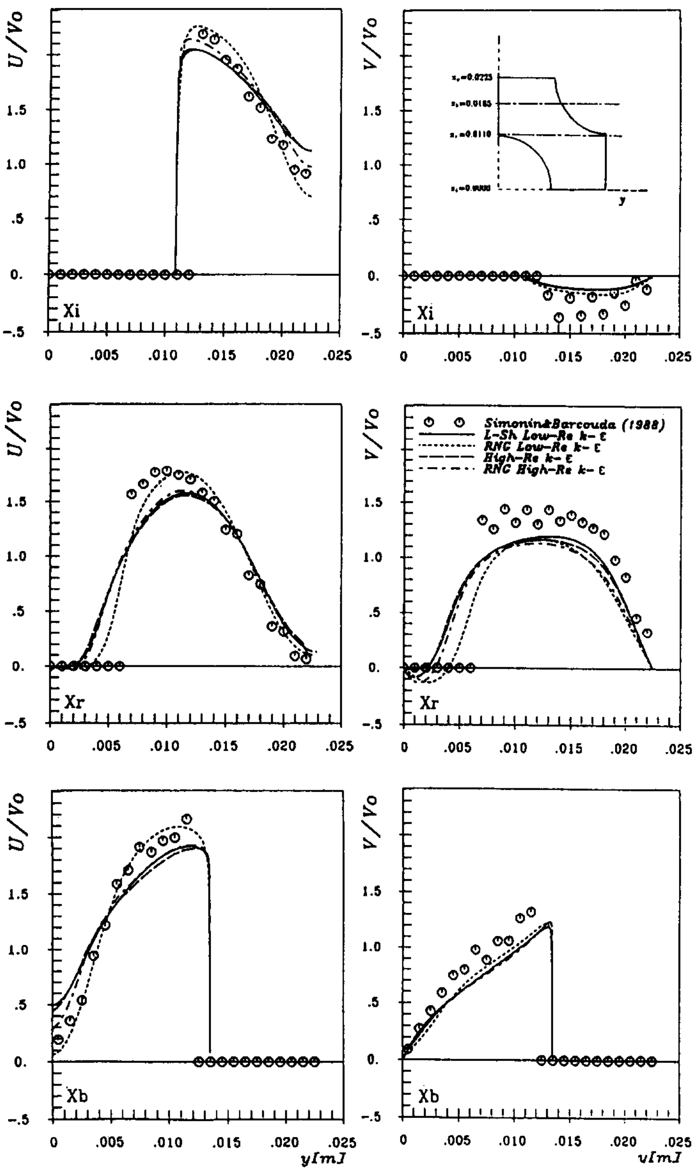

Figure 12 Mean velocities in the staggered tube bundle. ∘∘: experiments (Simonin and Barcouda 1988); computations by k–ε models (Hadžić, Hanjalić and Perić 1993)

proposed. In fact, some models, e.g., Lam and Bremhorst (1978) are preferred by some users because of their robustness, which is usually achieved by introducing empirical damping functions in terms of local wall distance y^+ (or $\mathrm{Re}_t^+ = yk^{1/2}/\nu$). These functions avoided the use of second velocity derivatives as in the ε-equation in the Jones–Launder model. A comparison of a number of models was given by Patel et al. (1985). Still, some recent tests of bypass transition (Savill 1993) as well as computation of some separated flows (Franke and Rodi 1991) give preference to the Jones–Launder–Sharma model. More recently, Rodi and Mansour (1993) analyzed several popular low-Re-number k–ε models with the aid of DNS data for channel flow from Mansour et al. (1988) and found that none of the considered models produced an adequate form of the damping function f_μ, used in the eddy-viscosity expression. They proposed a new f_μ in terms of y^+ as well as new modifications to the ε-equation.

The form of the "true" ε-equation compatible with the Re-stress model was first proposed by Hanjalić and Launder

(1976):

$$\frac{\partial \varepsilon}{\partial t} + U_k \frac{\partial \varepsilon}{\partial x_k} = \frac{\partial}{\partial x_k}\left(\nu \frac{\partial \varepsilon}{\partial x_k}\right) + \frac{\partial}{\partial x_k}\left(C_\varepsilon \frac{k}{\varepsilon} \overline{u_k u_l} \frac{\partial \varepsilon}{\partial x_l}\right) + C_{\varepsilon 1} P \frac{\varepsilon}{k} - C_{\varepsilon 2} f_\varepsilon \frac{\varepsilon \tilde{\varepsilon}}{k} + C_{\varepsilon 3}\, \nu \frac{k}{\varepsilon} \overline{u_j u_k} \frac{\partial^2 U_i}{\partial x_j \partial x_l} \cdot \frac{\partial^2 U_i}{\partial x_k \partial x_l} \tag{25}$$

Like the model of Jones and Launder, it contains an extra term with a second velocity derivative whose origin can be traced to the exact transport equation for ε, and one scalar function in terms of turbulence Re number, $f_\varepsilon = 1 - (0.4/1.8) \exp(-(\mathrm{Re}_t/6)^2)$, that ensures a correct switch from the initial to the final period of decay of isotropic turbulence.

In spite of some obvious shortcomings, the equation performed reasonably well in a variety of thin shear flows. It also served as a basis for further modifications, which in most cases were directed towards the replacement of the computationally inconvenient second velocity derivative with some empirical functions. Due to the lack of reliable experimental data for the energy dissipation, no verification of the ε-equation, in particular for the near-wall region, was possible until the appearance of DNS data a few years ago, which revealed that none of the current models could reproduce the near-wall behavior of ε in accord with direct numerical simulation (see also So et al. 1991). The standard models, both within the two-equation and Re-stress framework, predict a peak in ε at $y^+ = 10$, whereas DNS gave the maximum ε at the wall (Figure 13).

The discovery of this shortcoming stimulated the recent appearance of a number of new proposals for the modification of Equation 25. Some authors concentrate on the ε-equation only (Rodi and Mansour 1993), whereas others consider general modifications of the model as a whole, so that it is difficult to distinguish individual effects of the introduced changes. Shima (1988) abandoned the last term in Equation 25, but introduced a new term as a function of ε and $\tilde{\varepsilon} = \varepsilon - \nu \partial^2 k / \partial x_l \partial x_l$, supposedly to compensate for the failure of Equation 25 to satisfy the "coincidence" of $\partial\varepsilon/\partial t$ and $\partial/\partial t(\nu \partial^2 k / \partial x_l \partial x_l)$ at the wall. In addition, Shirma replaced the $C_{\varepsilon 1}$ by a function of P/ε and introduced a new damping function in terms of y^+. A test of the model in a fully developed, low-Re-number pipe flow produced only slight improvements of the stress components close to the wall. The model of Launder and Shima (1989) employs the ε equation in its basic, high-Re-number form (Equation 20) but with the coefficient $C_{\varepsilon 1}$ replaced by $(C_{\varepsilon 1} + \Psi_1 + \Psi_2)$, where $\Psi_1 = 2.5\, A(P/\varepsilon - 1)$ and $\Psi_2 = 0{,}3(1 - 0.3\, A_2) \exp[-(0.002\, \mathrm{Re}_t)^2]$. With their modification of the pressure-strain term in which all coefficients are treated as function of stress-anisotropy invariants and of Re_t, they obtained satisfactory predictions of several boundary-layer-type flows at both favorable and adverse pressure gradients. More recently, Shima (1991, 1993) also reported a reasonable success in reproducing 3-D boundary layers and oscillating 2-D boundary layers, indicating that the model, in spite of some formal shortcomings, seems robust and potentially useful for application in more complex flows. It should be pointed out, however, that the predicted shape of the dissipation rate close to the wall retained almost the identical form as obtained by most other models with a peak away from the wall. A more serious deficiency is its performance at high acceleration: in a sink flow, the model leads to a collapse of turbulence at a considerably lower value of the acceleration parameter than experimentally established. Lai and So (1990) and So, Zhang, and Speziale (1991) modified further the "coincidence" term of Shima (1988) and introduced some new damping functions in terms of the channel/pipe mean-flow Re number. In conjunction with their modification of the pressure-strain term, Lai and So (1990) obtained some improvements, though still insufficient, of the channel- and pipe-flow predictions at low Re numbers. Within the framework of their new low-Re-number, second-moment model, Launder and Tselepidakis (1991) use essentially the form of the ε equation as given by Equation 25, but with $C_{\varepsilon 1} = 1$ and $C_{\varepsilon 2} = 1.92/[1 + 0.63\,(AA_2)^{1/2}]$ and with an additional diffusive term $D_\varepsilon^p = \partial/\partial x_k (C_{\varepsilon 4} \nu (\tilde{\varepsilon}/k) \partial k / \partial x_k)$, supposed to model the pressure diffusion. Excellent reproduction of all stress components in channel flows at two low Re numbers are partially accomplished by this new form of the ε equation, although the major effect seems to be achieved by the modifications of the stress equation, as seen by the still unsatisfactory prediction of ε and of the overall stress budget.

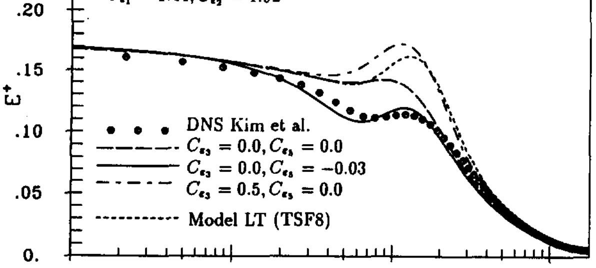

Figure 13 Computed ε from Equation 25 with additional term $C_{\varepsilon 5}(\partial k/\partial y)^2$; $\overline{u_i u_j}(y)$ and $U(y)$ data from DNS

Apart from the model of Launder and Shima (1989), all other models were mainly tested in simple plane channel or pipe flows. It is symptomatic that for these simple equilibrium flows, most of the models gave reasonable results for the mean-flow parameters and turbulent stresses, in spite of often very unsatisfactory budgets of stress components. The deficiencies of the ε-equation and a wrong shape of ε were compensated for by deficiencies in other parts of the models, so that the choice of the ε-equation seemed to be more a matter of taste or of computational convenience. Some new efforts concentrate solely on the improvement of the form of the dissipation equation in order to reproduce the behavior of ε in better agreement with the DNS results. Rodi and Mansour (1993) applied scaling argument to analyze the term-by-term modeling of the exact ε-equation (Equation 20) and employed the ratio of the time scale of the mean rate of strain S to the Kolmogorov time scale of dissipative motion $(\nu/\varepsilon)^{1/2}$ to define the damping function for the coefficient $C_{\varepsilon 2}$, i.e., $f_3 = \exp(2R_p^3)$, where $R_p = -\overline{u_1 u_2}/(0.3k)(S/\varepsilon/\nu)^{1/2}$, by which they model jointly $P_{\varepsilon 1} + P_{\varepsilon 2} - Y$. Although they reproduced the modeled group well in accord with the DNS results, it is pertinent to note that this modification will not satisfy, e.g., the decay of isotropic turbulence in the final period at low Re numbers. They also introduced an extra term in the model of $P_{\varepsilon 3}$ that involves $(\partial k/\partial y)$, $(\partial U/\partial y)$, and $(\partial^2 U/\partial y^2)$. The authors did not report any testing of the new model as yet.

Other routes are also possible. Hanjalić and colleagues have recently tested effects of the addition of various terms, in invariant forms, to Equation 25. Some can be traced in the exact equation for ε. Several terms individually or in combination can bring the ε-profile in full accord with the DNS data (Hanjalic 'and Jakirlić 1992), as shown in Figure 13. However, it should be recalled that deficiencies in ε equations of current models must be reflected in deficiencies in other equations which compensate each other, while producing acceptable predictions of tested flows. Any substantial modification of the ε-equation requires inevitably a substantial modification of the $\overline{u_iu_j}$-equation (in particular of the Φ_{ij} and D^t_{ij} terms) if the model is expected to reproduce the practical flows in similar agreement to the current models. The idea, which essentially implies a separate modeling and verifying of each term in the model transport equations, as shown by Rodi and Mansour (1993) for the ε equation in the k–ε model, is also worth exploring for the Re-stress model, but represents a major task that still has not been carried out to full extent.

Some recent developments indicate a possibility to design the transport equations applicable in low-Re-number flows and up to the wall without employing any empirical damping functions. The secret seems to be in the use of the Kolmogorov time scale (Durbin 1991, 1993) if the turbulence Re number becomes sufficiently low to incur the overlapping of the energy-containing and dissipative parts of the spectrum, as happens very close to the wall. Durbin argues that the eddy time scale cannot be smaller than the Kolmogorov scale and proposed to use the expression

$$\tau = \max\left[\frac{k}{\varepsilon}, C_\tau\left(\frac{\nu}{\varepsilon}\right)^{1/2}\right] \tag{26}$$

The argument sounds correct, except for the need to introduce the empirical constant $C_\tau \approx 6.4$, which brings in a dose of arbitrariness.[5] Although the expression becomes effective only for $y^+ < 5$ (with an increase in C_τ, this range can be extended), in conjunction with the eddy viscosity defined as $\nu^t_{ij} = C_\mu\overline{u_iu_j}\tau$ (with $C_\mu = 0.23$ to account for the ratio $\overline{u_2^2}/k$), the use of the new time scale eliminated the need to employ the damping function f_μ. Durbin employs the same time scale instead of k/ε also in both terms of the dissipation Equation 25, as well as in the model of the return-to-isotropy pressure strain term $\Phi_{ij,1}$. An analogous approach was adopted for defining the turbulence length scale, where needed. In conjunction with his elliptic relaxation model of the pressure-strain term, Durbin obtained turbulent stresses in a channel flow at Re = 5600 in very good agreement with the DNS data. Good predictions were obtained also for constant pressure and adverse pressure gradient boundary layer, and somewhat poorer predictions of the boundary layer on a convex curved surface.

In fact, if the expression for the eddy viscosity in the RNG variant of low-Re-number k–ε model is interpreted as that for the total (effective) viscosity $\nu_{\text{eff}} = \nu + \nu_t^6$, then

$$\nu_{\text{eff}} = \nu\left[1 + \sqrt{\frac{C_\mu}{\nu}}\frac{k}{\sqrt{\varepsilon}}\right]^2 \tag{27}$$

yields the eddy viscosity $\nu_t = C_\mu k\tau$ (with $C_\mu = 0.084$) in which the time scale τ is given as the sum of the time scale of energy-containing eddies and of the Kolmogorov time scale, i.e.,

$$\tau = \frac{2}{\sqrt{C_\mu}}\left(\frac{\nu}{\varepsilon}\right)^{1/2} + \frac{k}{\varepsilon} \tag{28}$$

where $2/\sqrt{C_\mu} = 6.9$ is close to the value used by Durbin. This may explain the relative success of the RNG-derived k–ε model in the near-wall region without using any damping function.

4.2.1. Dissipation-rate tensor in the $\overline{u_iu_j}$-equation.

Most Re-stress models regard the dissipation tensor ε_{ij} as isotropic even at relatively low turbulence Re numbers, so the most frequent form of the model—at least for high-Re-number flows—is $\varepsilon_{ij} = 2/3\varepsilon\delta_{ij}$. Equipartition of dissipation among the normal stress components and neglect of the viscous terms in the shear-stress equation at sufficiently high-turbulence Re number has been regarded as justified, since this requirement is far weaker than complete isotropy of the dissipating eddies (Schwarz and Bradshaw 1992) and still weaker than the stress anisotropy. Even if this condition is not fully satisfied, any anisotropy can still be absorbed in the model of Φ_{ij}, as argued by Lumley (1978). The anisotropic form of ε_{ij} was generally adopted only for the region very close to a solid wall, and its degree has usually been related to the local stress anisotropy a_{ij}. The coupling is accomplished by aid of turbulence Re number Re_t. A common form, introduced by Hanjalić and Launder (1976), and frequently used for modeling low-Re-number and near-wall flows, that satisfies the limiting conditions at both high and low Re numbers is

$$\varepsilon_{ij} = (1 - f_s)\,\tfrac{2}{3}\delta_{ij}\varepsilon + f_s\varepsilon'_{ij} \tag{29}$$

where originally $\varepsilon'_{ij} = \varepsilon\overline{u_iu_j}/k$. Equation 29 expresses a proportionality of large-scale (stress) and small-scale (dissipation) anisotropies, i.e., $e_{ij} = f_s a_{ij}$, where $e_{ij} = \varepsilon_{ij}/\varepsilon - 2/3\delta_{ij}$. The empirical function $f_s(\text{Re}_t)$ should ensure a transition from one mode to another ("decoupling") at appropriate Re_t. Hanjalić and Launder (1976) formulated f_s to decay fast as the distance from the wall increases, yielding the isotropic ε_{ij} in the outer wall region even at relatively low bulk Re numbers. DNS data revealed a different trend: in a plane channel flow at Re = 5600 (Mansour et al. 1988) as well as for Re = 14,000 (Kim et al. 1987), over most of the wall region (up to $y^+ = 60$), the dissipation tensor exhibited a high degree of anisotropy very much in accord with $\varepsilon_{ij} = \overline{u_iu_j}/k\varepsilon$, although ε_{12} was small everywhere except very close to the wall ($y^+ < 10$). These findings seem to substantiate some arguments, proposed earlier, that the dissipation anisotropy in near-wall flows is dominantly caused by the strong wall influence, which permeates well outside the viscosity-affected region. For that reason, but also to gain more flexibility, several groups of modelers introduced the second and third invariants A_2 and A_2, as well as the "flatness" parameter A (Launder and Tselepidakis 1991; Gilbert and Kleiser 1991).

Hanjalić and Jakirlić (1992) argued that neither Re_t nor A are individually suited to model ε_{ij} in accord with the DNS data. A comparison of Re_t for Re = 5600 and 14,000 shows that the turbulence Re numbers, Re_t, coincide reasonably well up to $y^+ = 10$ and then depart so that Re_t cannot be employed to generate a unique f_s up to $y^+ = 60$, as required by DNS data. On the other hand, DNS data show that the stress anisotropy invariants and the flatness parameter remain almost uninfluenced by the bulk Re number, whereas we expect that at very high Re numbers, ε_{ij} should become isotropic already at the edge of the viscous wall region. For small Re (e.g., Re = 5600) the function $f_s = 1 - \sqrt{A}$ (Gilbert and Kleiser 1991) produced ε_{ij} close to the DNS data—apart from the limiting near-wall behavior. However, the function will produce the same degree of anisotropy in ε_{ij} at very high bulk Re number, contrary to real flows, where ε_{ij} is expected to be fully isotropic. The function $f_s = \exp(-20A^2)$, as proposed by Launder and Tselepidakis (1991) satisfies the high Re number limit, but not the low Re number case (Re = 5600), since it decays sharply already at $y^+ = 30$. For these reasons, Hanjalić and Jakirlić (1992) proposed that f_s be modeled in terms of the flatness parameter of the small-scale motion $E = 1 - 9/8(E_2 - E_3)$, where $E_2 = e_{ij}e_{ji}$ and $E_3 = e_{ij}e_{jk}e_{ki}$ are the second and third invariants of the stress dissipation rate tensor $e_{ij} = \varepsilon_{ij}/\varepsilon - 2/3\delta_{ij}$, respectively. The expression in the form

$$f_s = 1 - E^n$$

satisfies the condition of local isotropy of the small-scale motion at high Re numbers, where $E = 1$ and ε_{ij} becomes isotropic irrespective of a_{ij}. It also satisfies the two-component limit where $E = 0$ and $e_{ii} = a_{ii}$ (no summation of indices) for the remaining components. We tested several forms by using the DNS results for ε and $\overline{u_i u_j}$ and found that $f_s = 1 - E^4$ reproduced ε_{ij} in very good agreement with DNS data for ε_{ij} for all three normal components, although failed to bring sufficient improvement in ε_{12} (Figure 14). Still better results were achieved with a modified function $f_s = 1 - A^{1/2}E^4$, which does not fully satisfy the mentioned limits of small-scale isotropy, but proved to be computationally more robust. The dissipation anisotropy can now be expressed as

$$e_{ij} = a_{ij}(1 - A^{1/2}E^2) \tag{30}$$

which illustrates essentially the nonlinearity of the proposed model.

Launder and Reynolds (1982) noted that ε' and, consequently, Equation 29 does not satisfy the limiting wall values of the individual components of ε_{ij} for the case when i and/or j take the value 2 (corresponding to the normal-to-the-wall coordinate). Simple derivation from Equation 2 shows that at a solid wall or phase interface, $\varepsilon_{ij}/\overline{u_i u_j} = \varepsilon/k$ only if i and j take the values 1 or 3, while $\varepsilon_{22}/\overline{u_2^2} = 4\varepsilon/k$ and $\varepsilon_{12}/\overline{u_1 u_2} = 2\varepsilon/k$. They proposed an expression in a general form that satisfies the limiting wall values of ε_{ij}. Subsequent correction by Kebede, Launder, and Younis (1985) ensured that the sum of the diagonal components contracts to ε and also reduced a prolonged effect of wall correction.

Hanjalić and Jakirlić (1992) concluded that the wall corrections of ε_{22} and ε_{12}, even in the corrected form, pertain still too far away outside the intended near-wall region and tested a modified form of the correction,

$$\varepsilon'_{ij} = \frac{\varepsilon}{k}\,[\overline{u_i u_j} + (\overline{u_i u_k} n_k n_j + \overline{u_j u_k} n_k n_i + \delta_{ij} \overline{u_k u_l} n_k n_l) f_d] \Big/ \left(1 + 1.5\,\frac{\overline{u_p u_q}}{k}\, n_p n_q f_d\right) \tag{31}$$

which differs from the original expression by the introduction of a damping function $f_d = (1 + 0.1\mathrm{Re}_t)^{-1}$ (used earlier in the expression for ε_{ij}). This modification was employed together with the new function f_s to produce results shown in Figure 14.

A nonlinear relationship between e_{ij} and a_{ij} was recently proposed by Hallbäck, Groth, and Johansson (1990) based on the expansion of the expression $e_{ij} = f a_{ij} + g(a_{ik}a_{jk} - 1/3 A_2 \delta_{ij})$, where f and g are assumed to be functions of stress invariants A_2 and A_3, the mean rate of strain and mean vorticity. By imposing the symmetry conditions, zero trace, and Caley–Hamilton relation for the a_{ij} tensor, and truncating at the third-order terms, the expression reduces to

$$e_{ij} = a_{ij}[1 + \alpha(\tfrac{1}{2} A_2 - \tfrac{2}{3})] - \alpha(a_{ik}a_{jk} - \tfrac{1}{3} A_2 \delta_{ij}) \tag{32}$$

With $\alpha = 3/4$, the model reproduced well several sets of DNS results for homogeneous turbulence, although Jovanović, Ya, and Durst (1992) reported very modest success in reproducing the DNS results for ε_{ij} components in a plane channel flows.

Perhaps we should expect in the near future the use of the modeled differential transport equations for each component of the dissipation-rate tensor, as recently proposed by Tagawa et al. (1991). Alternatively, for more complex flows, the idea of jointly modeling the return-to-isotropy and stress dissipation rate, $\Phi_{ij} - \varepsilon_{ij}$, as advocated by some authors (e.g., Lumley and Newman 1977), may come again into focus, in spite of the known fact that these two terms represent very different processes.

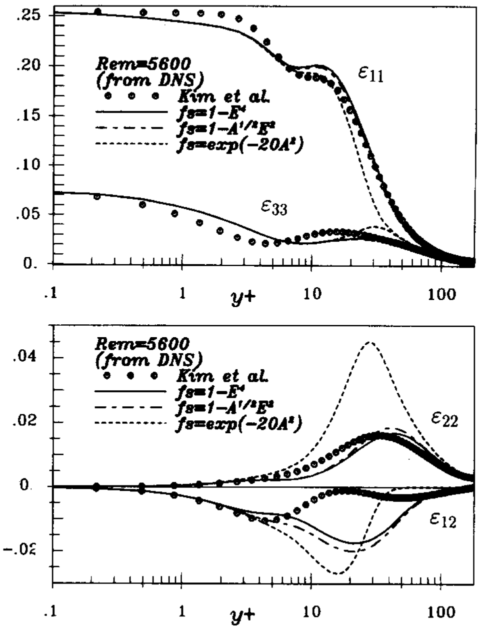

Figure 14 Components of ε_{ij} computed from DNS data for ε and $\overline{u_i u_j}$ for different f_s

4.3. Physical constraints and industrial standards

Much was said about various constraints imposed upon the models by exact mathematical derivation and physical arguments, many of which cannot be fulfilled by any of the current models. Yet most of the models reproduce acceptably well the experimental or DNS data for a considered class of flows. Either the constraints are not encountered in the considered flows, or deficiencies in various segments of the model compensate for each other. A typical example is the plane channel flow (and other thin shear flows) where most models reproduce well the mean-flow properties and second moments, but fail to reproduce the budget of individual stresses, and even of the kinetic energy. Figure 15 shows the predictions of the turbulent stresses for a plane channel flow at Re = 5600 obtained by the recent model of Launder and Tselepidakis (1991) (denoted as LT) already described (but without the new reflection term), and by the model of Launder and Shima (1989) (denoted as LS).

Also presented are the results obtained by a recent variant of the model tested by Hanjalić and colleagues (Hanjalić, Jakirlić, and Hadžić 1993; Jakirlić, Hanjalić, and Durst 1993). This model (denoted as HJ) employs the dissipation equation in the form of Equation 25 with $C_{\varepsilon 3} = 0.25$, and models ε_{ij} as described earlier by Equations 29 and 31, whereas the standard high-Re-number form of the linear model of Φ_{ij} was adopted with coefficients taken as functions of A_2, A_3, and Re_t, i.e.,

$$C_1 = C + A^{1/2}E^2, \qquad C_2 = 0.8A^{1/2}, \qquad C_1^w = 1 - 0.7C,$$

$$C_2^w = \min(A, 0.3)$$

where

$$C = 2.5AF^{1/4}f, \quad F = \min(0.6, A_2), \quad f = \min\left[\left(\frac{\mathrm{Re}_t}{150}\right)^{3/2}, 1.0\right]$$

These functions are purely empirical with a little physical justification. This cannot be avoided with the assumed linear form of the basic high-Re-number model of Φ_{ij}. However, they are all expressed in terms of scalar parameters and are independent of the wall configuration. They also satisfy basic constraints relevant to near-wall flows such as the 2-D turbulence state and the limits of vanishing and very high Re_t.

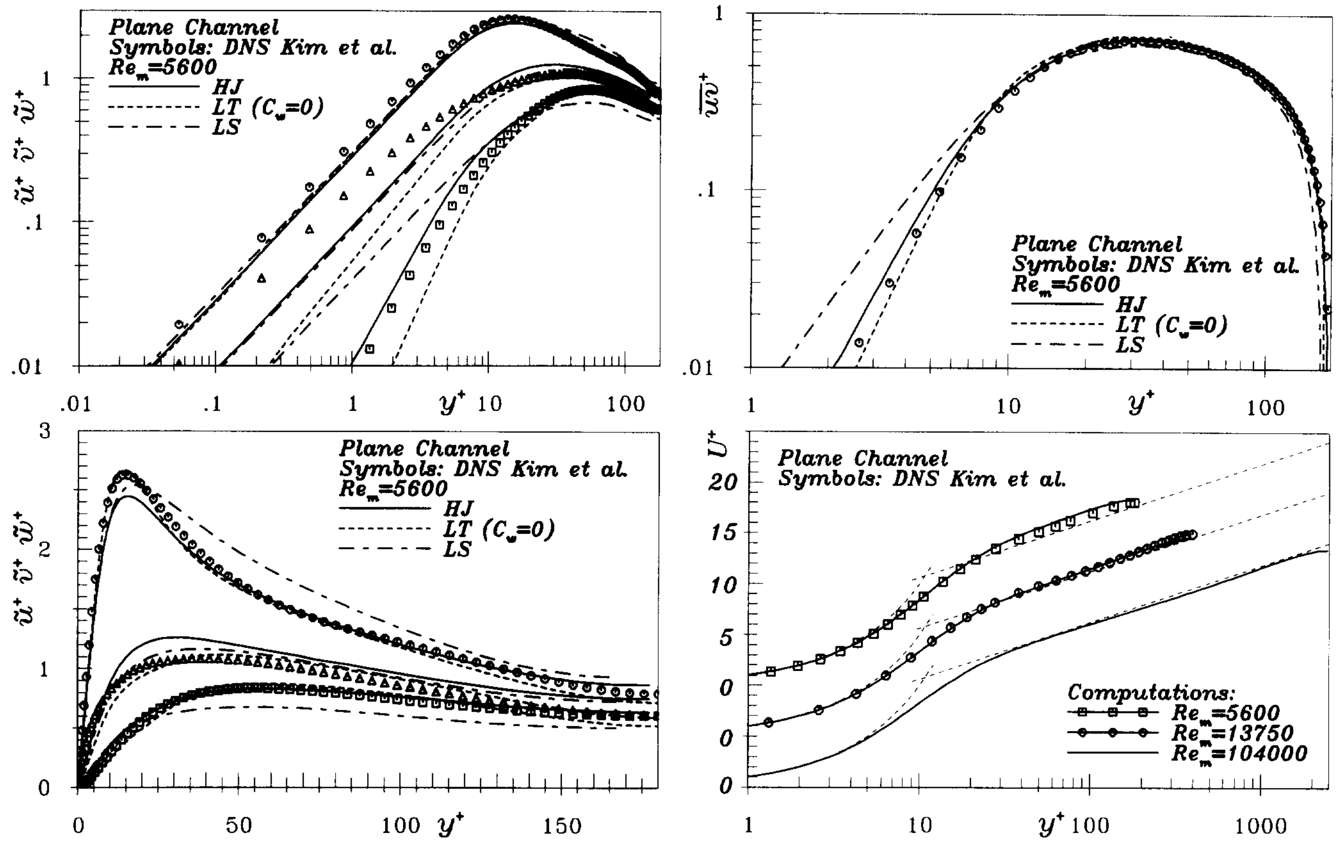

Figure 15 Turbulent stresses and mean velocity in a plane channel. Symbols: DNS (Kim et al. 1987); lines: computations; LT: Launder and Tselepidakis 1991; LS: Launder and Shima 1989; HJ: Hanjalić and Jakirlić (unpublished)

As seen, the LT and HJ models gave satisfactory reproduction of all stress components, with appropriate slopes and values (apart from a descrepancy in $\overline{u_3^2}$ very close to the wall). A visibly poorer agreement was obtained with the LS model. For illustration, the logarithmic velocity profiles for three Re numbers are also presented. The computed stress budget for a constant-pressure boundary layer, shown in Figure 16, is in a reasonable agreement with the DNS results of Spalart (1988), apart from very close to the wall. A very similar agreement with the DNS data is obtained for a plane channel. It should be noted that the modeled Φ_{ij} is identified with $\Pi_{ij} = \Phi_{ij} + D_{ij}^p$ (since the pressure diffusion is not modeled explicitly) and compared with the DNS results for Π_{ij}.

In addition to a desired computational sturdiness, the model has shown a high degree of generality, reproducing also satisfactory several more complex 2-D and 3-D thin shear flows. Examples include flows with severe favorable (including the case with laminarization) and adverse pressure gradient, oscillating flow around a zero mean with rapid variation of the mean-flow velocity, as well as some 3-D shear-induced flows with zero and favorable stream-wise pressure gradient, as illustrated in Figures 17 to 20. It should be noted that the inclusion of the term $C_{\varepsilon 4}\overline{u_i u_l}(\partial U_i/\partial x_l)$ (with $C_{\varepsilon 4} \approx 2.4$), which enhances the production of ε by irrotational straining, as proposed in Hanjalić and Launder (1980) (the approximation of the effect of the third term in Equation 21 for thin shear flows), contributed to the prediction of laminarization at the appropriate value of the acceleration parameter $K = (\nu/U_e^2)(dU_e/dx) \approx 3.2 \times 10^{-6}$ and also improved the predictions of flows at adverse pressure gradient. Together with a number of examples reported by Launder and Shima (1989) and Shima (1991, 1993) with a similar type of model, the illustrations show that these and similar models display a considerable degree of generality and may be employed for industrial computations. In spite of the lack of verification in more complex 3-D flows, still-present ad hoc empiricism, and the inability to fulfill all mathematical and physical constraints, they can be regarded as less uncertain than the currently most frequently employed k–ε or similar two-equation eddy-viscosity models. Of course, the use of more sophisticated, higher-order models, implementation of multiscale concept and other new ideas discussed earlier as a basis on which the low-Re-number and wall vicinity modifications are built, may lead to less empiricism and higher generality, particularly if complex separating flows are to be predicted. Models of such universality, however, do not seem to be yet in the offing.

5. Possible future directions

5.1. Turbulence scales

Although the turbulence scales are primarily defined to characterize the turbulent mixing through the definition of eddy viscosity at a point in the flow, they are also utilized to model all superfluous terms in the transport equations for turbulence parameters. Since these terms represent physically different turbulence interactions that are known to occur at different rates, the use of a single time-and-length scale is one of the basic approximations pertinent to all single-scale models, the abandonment of which represents an unexploited source of refinements of the model. In addition to employing a separate scale for the low-Re-number region near the wall, as suggested by Durbin (1991, 1993) and implied by the RNG models of Yakhot and Orszag (1986), two other concepts deserve

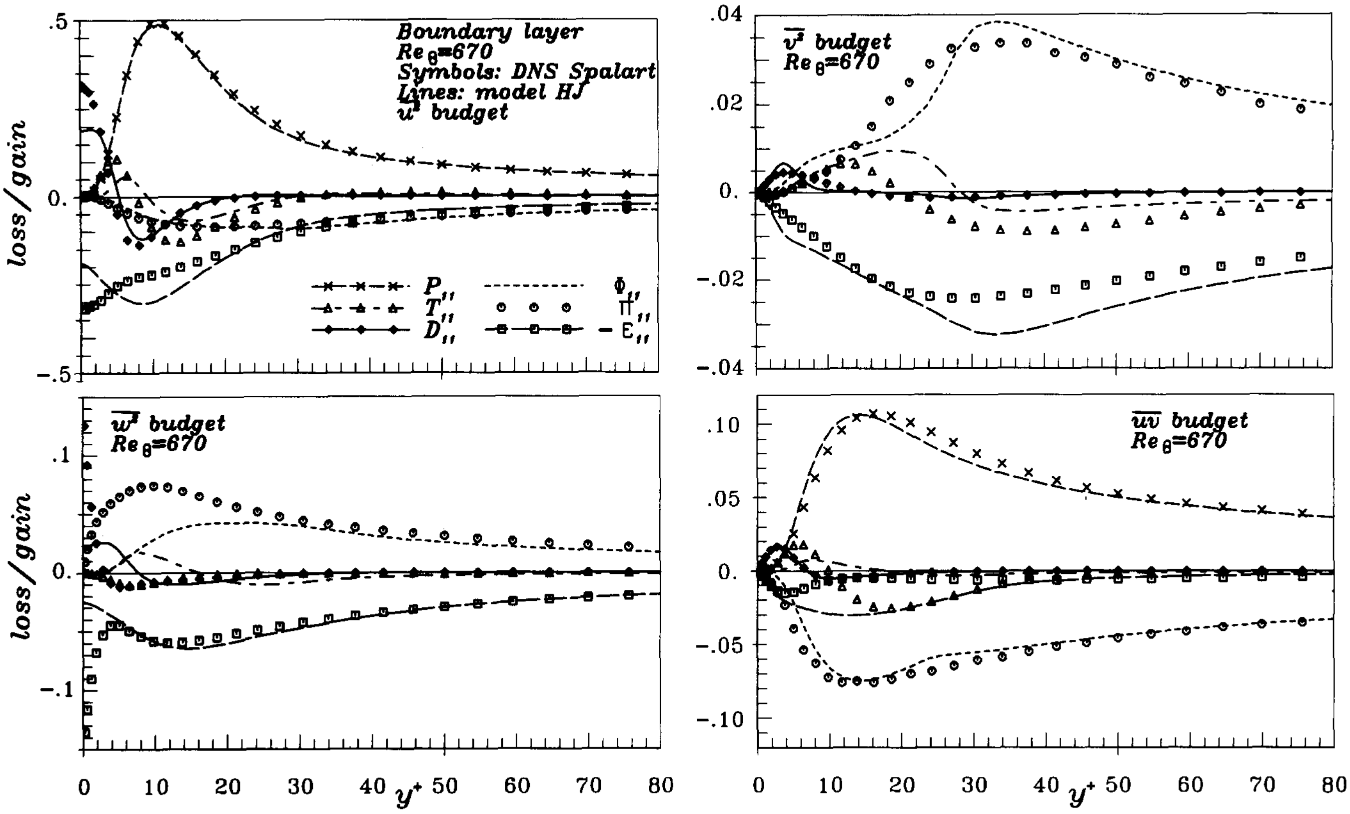

Figure 16 Budgets of turbulent stresses in a constant pressure turbulent boundary layer. Symbols: DNS (Spalart 1988); lines: computations (Hanjalić and Jakirlić unpublished)

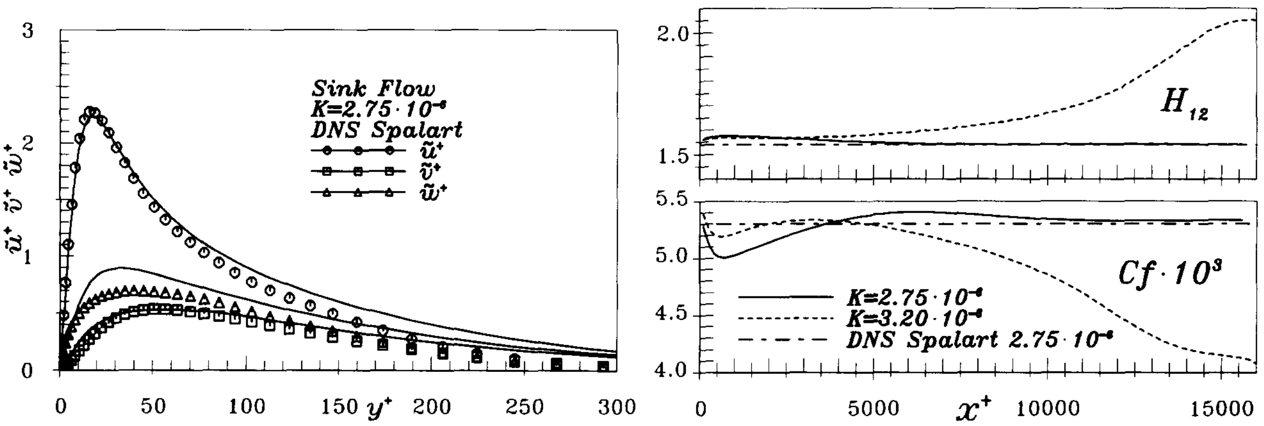

Figure 17 Turbulent stresses, friction coefficient, and shape factor in a sink flow at $K = 2.75 \times 10^{-6}$. Symbols: DNS (Spalart 1988); lines: computations (Jakirlić et al. 1993). (Note: dotted lines show laminarization at $K = 3.2 \times 10^{-6}$)

attention: the split-spectrum multiple-scale concept and the tensorial-scale model.

The split-spectrum multiple-scale concept. This concept, proposed by Hanjalić, Launder, and Schiestel (1980) and Schiestel (1983), offers an intriguing possibility to construct a model that employs two (or more) independently calculated turbulence time scales with which to characterize the dynamics of different turbulent interactions. The idea rests on the division of turbulence spectra into two parts that respond at different rates and in different ways to changes in the environment. As such, the concept may be regarded physically as an intermediate level of modeling between the single-point and two-point closure schemes, but computationally only marginally more laborious than single-point models. The spectrum is divided at the wave number, above which no significant mean-strain production occurs, i.e., the low wave number (production) region and the higher wave number (energy transfer and dissipation) region. Separate transport equations are to be modeled and solved for the Re-stress components (or for the turbulence energy in two-equation models) for each part of the wave number spectra. The transport equations are also solved for the energy transfer rate out of the production range and through the inertial subrange, the latter being assumed to be equal to the dissipation rate, ε.

The computational results of a number of notorious nonequilibrium homogeneous, free and wall thin shear flows

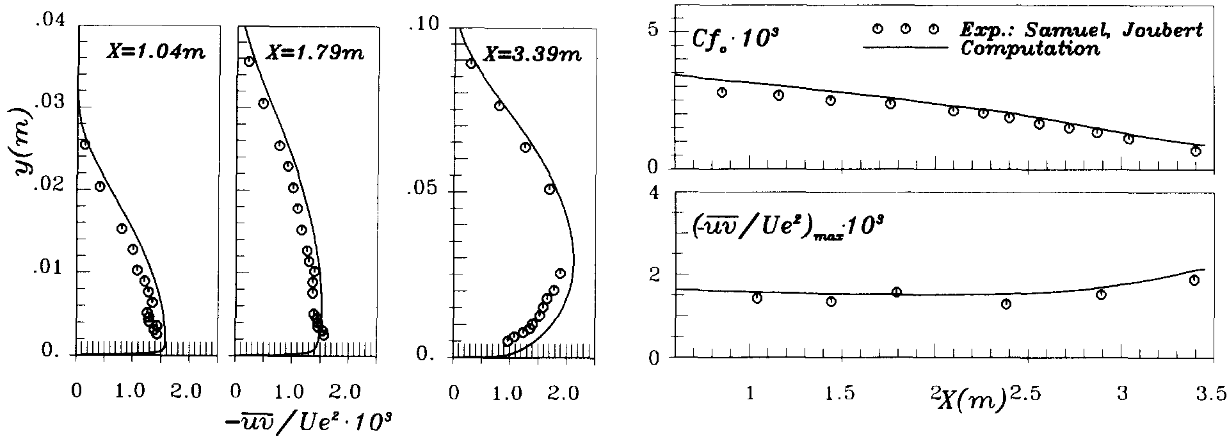

Figure 18 Turbulent shear stress profiles at selected stations, friction factor, and maximum $\overline{uv}$ in a boundary layer at adverse pressure gradient. Symbols: experiments (Samuel and Joubert); lines: computations (Hanjalić and Jakirlić unpublished)

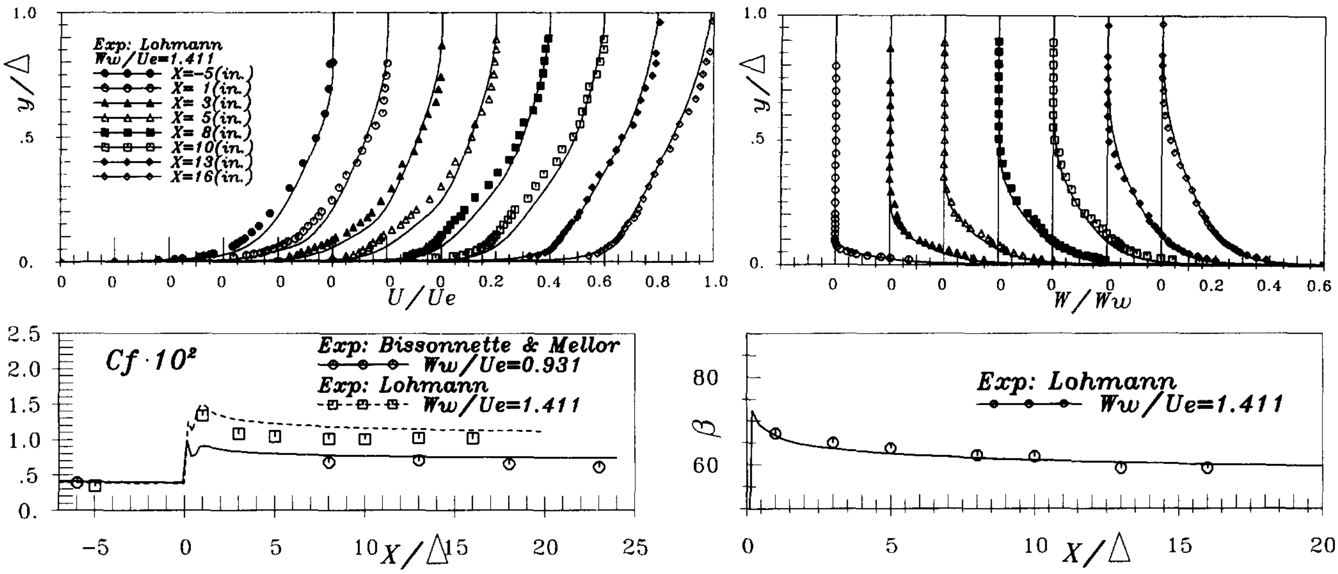

Figure 19 Streamwise and spanwise mean velocities, total friction coefficient, and angle between the wall shear stress and free-stream velocity in a 3-D boundary layer at a rotating aft cylinder. Symbols: experiments (Lohmann 1976); lines: computations (Jakirlić et al. 1993)

showed significantly closer agreement with experiments in comparison with single-scale-model predictions. This encouraging outcome was not pursued much further, perhaps because of the intimidating task of retesting a vast number of flows and tuning a number of new coefficients that inevitably appear as a consequence of the increased model diversity. Direct numerical simulation offers possibilities to get information on the terms in the budget of transport equations associated with each part of the spectrum for the adopted partitioning wave number(s) and to evaluate with more certainty the unknown coefficients. The idea is certainly appealing, and we may expect some further developments along these lines.

Tensorial length scale in eddy-viscosity models. Another interesting idea, proposed many years ago (Monin, in Monin and Yaglom 1975), but not exploited much for obvious reasons, is the stress-strain relationship that employs a length scale in the form of a symmetric tensor of the second rank, l_{ij} (the scale tensor), namely,

$$\overline{u_iu_j} = 2/3k\delta_{ij} - k^{1/2}(l_{ik}S_{kj} + l_{jk}S_{ki}) \tag{33}$$

This approach produces a nonisotropic eddy viscosity, regarded by some researchers as unavoidable for more accurate computation of 3-D flows (see, e.g., Bradshaw 1987; Cousteix 1986; Lakshminarayana 1986) for which the use of any model of order higher than the two-equation model is still formidably impractical. The obvious drawback is a need to define the ellipsoid of the length scales l_{ij}. A way to incorporate this idea into the currently popular models might be to keep the scalar length $l = k^{3/2}/\varepsilon$ as the spherical average of l_{ij} and to introduce the deviatoric part of it, γ_{ij}, which should account for the length-scale directional orientation, i.e., $l_{ij} = l\gamma_{ij}$. The deviatoric can be specified by an algebraic relationship like that proposed by Naot et al. (see Wolfshtein et al. 1975):

$$\gamma_{ij} = (1 - \Lambda)\delta_{ij}/3 + \Lambda\overline{u_iu_j}/(2k) \tag{34}$$

where Λ is a coefficient that can be evaluated on the basis of ratios of macroscales of the quasi-isotropic correlation functions (based on plane channel flow data Naot suggested to be 0.3 to 0.4). Close to a solid wall, the expression will produce

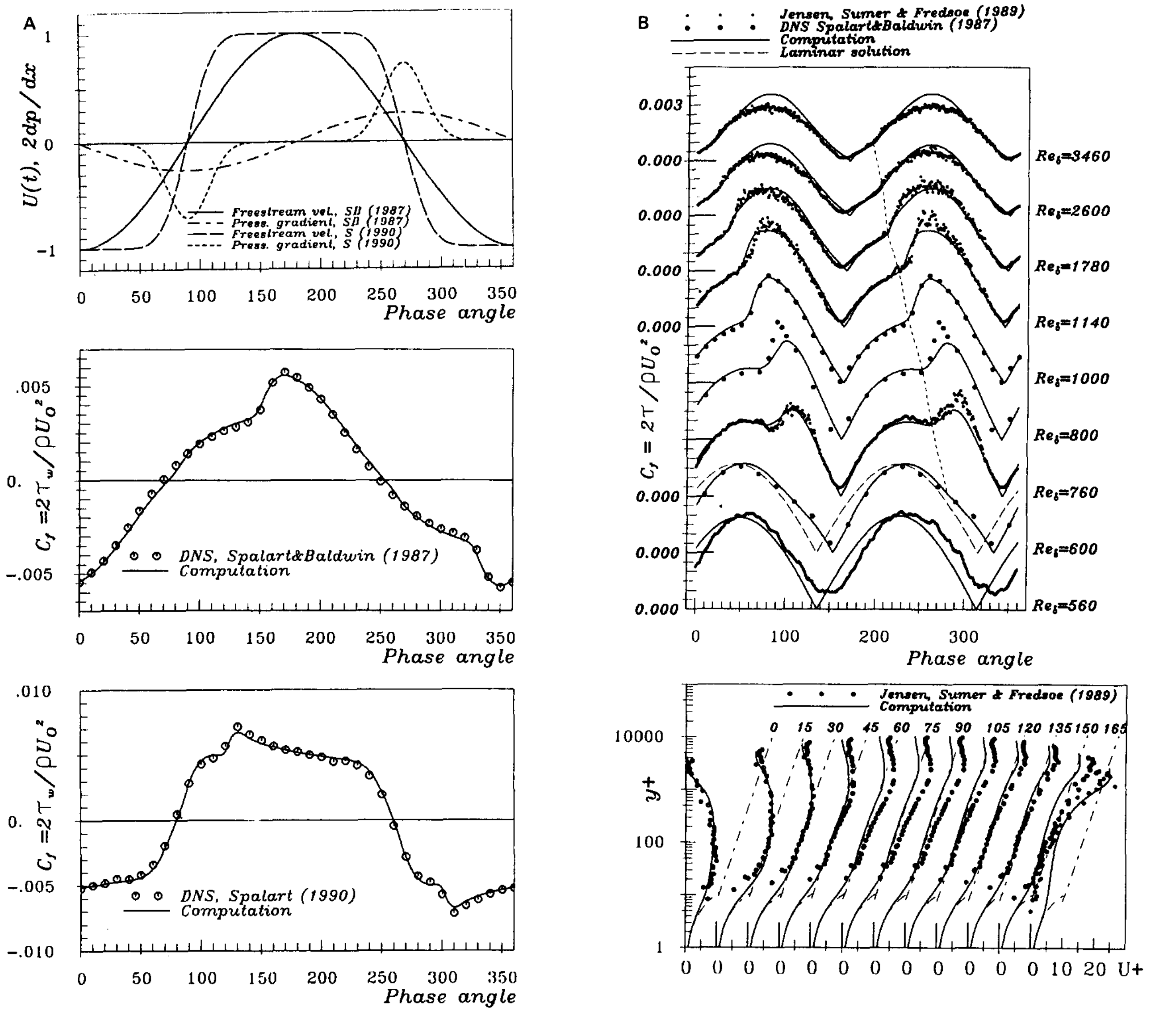

Figure 20 Oscillating boundary layer. (a) Evolution of wall shear stress at $Re_\delta = 1000$ for "mild" and "steep" variation of the free-stream velocity; (b) upper: evolution of shear stress for different Re numbers; lower: logarithmic velocity profiles at $Re_\delta = 33,460$ (lines: computations, Hanjalić et al. 1993)

$l_{22} < l_{33} < l_{11}$, as expected. This approach requires, however, a mode to express the ratio $\overline{u_iu_j}/k$ that calls for some kind of ASM model. The idea of a tensorial scale may, however, be easily incorporated into the full DSM or ASM.

5.2. Nonlinear stress-strain relationship

The recognized similarities between the mean turbulent flow of a Newtonian fluid and the laminar flow of viscoelastic fluids has frequently served in the past as an inspiration to extend the validity of simple models by the asymptotic expansion of the basic stress-strain relationship and inclusion of higher-order terms (e.g., Lumley 1978). Following that route in a consistent manner, and satisfying most of the mathematical constrains, Speziale (1987) extended the standard k–ε (and k–L) model by adding two more terms into the stress-strain expression, which are quadratic in the mean-velocity gradients—analogous to the viscoelastic terms in the second-order model of a Rivlin–Ericsen fluid. Such a nonlinear eddy-viscosity model incorporates a trace of memory of the mean-flow deformation into the expression for the Reynolds stresses:

$$\overline{u_iu_j} = 2/3k\delta_{ij} - k^{1/2}lS_{ij} + C_D l^2[(S_{im}S_{mj} - S_{mn}S_{mn}\delta_{ij}/3) + (\mathring{S}_{ij} - \mathring{S}_{mm}\delta_{ij}/3)] \quad (35)$$

where $\mathring{S}_{ij}$ is the Oldroyd frame-indifferent derivative of S_{ij},

$$\mathring{S}_{ij} = \frac{\partial S_{ij}}{\partial t} + U_k \frac{\partial S_{ij}}{\partial x_k} - S_{kj}\frac{\partial U_i}{\partial x_k} - S_{ki}\frac{\partial U_j}{\partial x_k} \quad (36)$$

and C_D is an empirical coefficient to which a value of 1.68 was assigned, associated with the adopted length scale $l = C_\mu k^{3/2}/\varepsilon$.

In the words of Speziale (1987), his model is a special case of a much more complex nonlinear eddy-viscosity model derived by Yoshizawa using Kraichnan's DIA formalism. Numerical tests of the model produced acceptable predictions of normal stresses in a fully developed plane channel (though not very satisfactory in the near-wall region) and of the secondary flow induced by a difference of normal Re-stresses in noncircular ducts (neither phenomena can be predicted with standard two-equation models). A substantial improvement of the prediction of the reattachment length of the recirculating zone behind a backward-facing step was also obtained.

Although this nonlinear model extends the range of validity of the standard models while maintaining most of their popular features (satisfying at the same time most mathematical constraints), like its linear antecedent, it violates the principle of physical coherence, since it expresses turbulence properties (here Re-stresses) as being directly proportional to the mean-flow properties (mean rate of strain). As such, the model cannot remedy some noted failures of the eddy-viscosity model, such as the noncoincidence of the positions of zero shear stress and mean rate of strain in asymmetric flows (a fact which may sound trivial from a practical point of view, but which is a sensitive indicator of a model's ability to mimic turbulent transport), not to mention local regions close to separation and reattachment points, corners and other singularities, and flows with history effects.

6. Concluding remarks

In spite of all the research efforts and substantial progress in developing more general simulation models, two-equation models still remain the most widely used fast engineering methods for computing complex turbulent flows. In comparison with second-order Re-stress models, they demand a more modest computational effort—an advantage that becomes progressively important in complex flows (three-dimensionality, irregular flow boundaries, multiphase flows, chemical reactions, etc.). Various remedies, designed to account for "extra effects," may improve predictions of complex flows, if employed knowledgeably.

There is, however, an evident disparity between the efforts on modeling refinements on the one hand and the use of models for calculation of complex flows on the other: very few industrial users take advantage of reported improvements—they seem to be generally satisfied with standard models with simple wall functions. Is it a question of distrust or insufficient understanding of the whole affair that leads to a blind use of whatever the simplest models offer?

Second-order closure models are still burdened with a high degree of uncertainty in modeling higher-order correlations due to a lack of sufficient knowledge of their physics. However, most model creators believe that the second-order closure models will be more in use in the future and that one can expect increasing efforts directed towards their refinement and generalization. Some new developments, discussed earlier, sound very promising, and a breakthrough may already be in the offing.

The successes in the development of modeling and computation techniques over the past 25 years have attracted many users among both the scientific and industrial communities. Some users even believe that they have at their disposal a tool powerful enough to calculate any complex flow. But, to quote Rotta's warning (1984), "... It should never be forgotten that the second order closure principles means a drastic simplification in the description of the turbulent motion. Very complex cases like the flow field around an arbitrarily shaped body ... will hardly be accessible by such methods. A really universal turbulence model is a dream and will be a dream possibly for ever."

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Notes

1. As compared with direct numerical simulation (DNS) or large-eddy simulation (LES), which take tens or hundreds of hours of supercomputer time.
2. Lumley (1978) argued that the pressure diffusion is typically about 20% of the velocity diffusion.
3. Recent analytical derivation of single-point models on the basis of the Renormalization group theory (RNG) suggests an increase of all diffusion coefficients in the transport equations by about 35%.
4. More recently, Smith and Reynolds (1992) showed that the RNG theory does not yield a production term of the form used in the standard model (Equation 20).
5. In fact, the Kolmogorov time scale has been indirectly invoked by most models that employed the turbulence Re number to modify the equations in the near-wall region, since $Re_t^{1/2} = k/\sqrt{\nu\varepsilon}$ is in fact the ratio of the energy-containing and dissipative time scales.
6. Presumably, this is what the authors meant, since in the limit of laminar flow they would have $\nu_t = \nu \neq 0$.

References

Amano, R. S. and Goel, P. 1986. Triple-velocity products in a channel with a backward-facing step. *AIAA J.*, **24** (6), 1040

Anderson, S. D. and Eaton, J. K. 1987. Reynolds shear stress development in pressure-driven three-dimensional turbulent boundary layer. *Proc. 7th Symp. Turbulent Shear Flows*, Toulouse, 13–1

Andree, J. C., de Moor, G., Lacarrere, P., Therry, G., and du Vachat, R. 1979. *Turbulent Shear Flows*, Vol. 1. Springer Verlag, Heidelberg

Atkinson, K. N. and Castro, I. P. 1991. Computation of separated turbulent boundary layers. *Proc. 8th Symp. Turbulent Shear Flows*, München, 20–2

Aupoix, B., Coesteix, J., and Liandrat, J. 1987. A way to derive the dissipation equation. *Turbulent Shear Flows*, Vol. 6. Springer, Berlin

Bardina, J. 1988. Turbulence modeling based on direct simulation of the Navier–Stokes equations. *Proc. 1st Natl. Fluid Dyn. Congress* (paper no. 88-3747-CP)

Bradshaw, P. 1972. The understanding and predictions of turbulent flow. *Aeronaut. J.*, **76** (739), 403

Bradshaw, P. 1987. Turbulent secondary flows. *Annu. Rev. Fluid Mech.*, **19**, 53–74

Bradshaw, P., Shabaka, I. M. A., and Mehta, R. D. 1982. Turbulent vortex flow. IC Aero TN 82-103, Imperial College, London

Bradshaw, P., Launder, B. E., and Lumley, J. L. 1991. Collaborative testing of turbulence models. *29th Aerospace Sciences Meeting*, Reno, NV (paper no. AIAA91-0215)

Choi, Y. D., Iacovides, H., and Launder, B. E. 1989. Numerical computation of turbulent flow in a square-sectioned 180 deg bend. *ASME J. Fluids Eng.*, **111**, 59–68

Chou, P. Y. 1945. On velocity correlations and the solution of the equations of turbulence fluctuation. *Q. J. Appl. Math.*, **3**, 38–54

Cormac, D. E., Leal, L. G., Seinfeld, J. H. 1978. *ASME J. Fluid Eng.*, **100**, 47–54

Cousteix J. 1986. Three-dimensional and unsteady boundary-layer computations. *Annu. Rev. Fluid Mech.*, **18**, 173–196

Cousteix, J. and Aupoix, B. 1981. Modelisation des equations aux tensions de Reynolds dans un repere en rotation. Unpublished report, Dept. d'Aerothermique, ONERA/CERT

Craft, T. J. and Launder, B. E. 1991. Computation of impinging flows using second-moment closures. *Eighth Symp. Turb. Shear Flows*, Munich, 8–5

Daly, B. J. and Harlow, F. H. 1970. Transport equations of turbulence. *Phys. Fluids*, **13**, 2634

Davidov, B. I. 1959. On statistical dynamics of incompressible turbulent fluid. *Dokl. Akad. Nauk SSSR*, **127**, 768 (translated in *Soviet Physics—Doklady*, **4**, 769)

Davidov, B. I. 1961. On statistical dynamics of incompressible turbulent fluid. *Dokl. Akad. Nauk SSSR*, **136**, 47 (translated in *Soviet Physics—Doklady*, **6**, 10)

Dekeyser, I. and Launder, B. E. 1984. A comparison of triple-moment temperature–velocity correlations in the asymmetric heated jet with alternative closure models. *Turbulent Shear Flows*, Vol. 4, L. J. S. Bradbury et al. (eds.). Springer, Berlin, 102

Demuren, A. O. and Rodi, W. 1984. Calculation of turbulence-driven secondary motion in non-circular ducts. *J. Fluid Mech.*, **140**, 189–222

Demuren, A. O. and Sarkar, S. 1993. Perspective: systematic study of Reynolds stress closure models in the computation of plane channel flows. *J. Fluids Eng.*, **115**, 5

Deng, G. B. and Piquet, J. 1991. k–ε turbulence model for low-Reynolds number wall-bounded shear flows. *Eighth Proc. Symp. Turb. Shear Flows*, Munich, 26–2

Devenport, W. J. and Simpson, R. L. 1990. The flow past a wing-body junction—an experimental evaluation of turbulence models. *Eighteenth Symp. on Naval Hydrodynamics*, Ann Arbor, MI

Dimaczek, G., Kessler, R., Martinuzzi, R., and Tropea, C. 1989. The flow over two-dimensional surface-mounted obstacles at high Reynolds numbers. *Proc. 7th Symp. Turbulent Shear Flows*, Stanford University, Stanford, CA, 10.1

Donaldson, C. du P., Sullivan, R. D., and Rosenbaum, H. 1972. Theoretical study of the generation of atmospheric clear air turbulence. *AIAA J.*, **10**, 162

Durao, D. F. G., Knittel, G., Pereira, J. C. F., and Rocha, J. M. P. 1993. Measurements and modelling of the turbulent near wake flow of a disc with a central jet. *Turbulent Shear Flows*, Vol. 8, F. Durst et al. (eds.). Springer, Berlin, 141–153

Durbin, P. A. 1991. Near-wall turbulence closure modeling without "damping functions." *Theor. Comp. Fluid Dynamics*, **3**, 1–13

Durbin, P. A. 1993. A Reynolds stress model for near-wall turbulence. *J. Fluid Mech.*, **249**, 465–498

Franke, R. and Rodi, W. 1991. Calculation of vortex shedding past a square cylinder with various turbulence models. *Eighth Symp. Turbulent Shear Flows*, Munich, 20–1

Fu, S., Huang, P. G., Launder, B. E., and Leschziner, M. A. 1986. A comparison of algebraic and differential second-moment closures for axissymmetric turbulent shear flows with and without swirl. Rep. TED/86/9, Mechanical Engineering Department, UMIST, Manchester, UK

Fu, S., Launder, B. E., and Tselepidakis, D. P. 1987. Accommodating the Effects of High Strain Rates in Modelling the Pressure-Strain Correlation. Rep. TED/87/5, Mechanical Engineering Department, UMIST, Manchester, UK

Gibson, M. M. and Launder, B. E. 1978. Ground effects on pressure fluctuations in the atmospheric boundary layer. *J. Fluid Mech.*, **86**, 491

Gibson, M. M. and Rodi, W. 1981. A Reynolds-stress closure model of turbulence applied to the calculation of a highly curved mixing layer. *J. Fluid Mech.*, **103**, 161–182

Gilbert, N. and Kleiser, L. 1991. Turbulence model testing with the aid of direct numerical simulation results. *Proc. 8th Symp. Turb. Shear Flows*, Munich, 26–1

Hah, C. 1981. Three-dimensional structures and turbulence closure of the wake developing in a wall shear layer. *AIAA 14th Fluid and Plasma Dynamics Conf.* (paper AIAA-81-1269). AIAA, Palo Alto, CA

Halbäck, M., Groth, J., and Johansson, A. V. 1990. An algebraic model for nonisotropic turbulent dissipation rate in Reynolds stress closure. *Phys. Fluids A (2)*, **10**, 1854–1866

Hanjalić, K. 1989. Practical predictions by single-point closure methods. *Proc. Z. Zaric Memorial Symp. on Near Wall Turbulence*, S. J. Kline (ed.). Hemisphere, Washington, DC

Hanjalić, K. and Jakirlić, S. 1992. A model of stress dissipation in second-moment closures. *Proc. 4th Eur. Turbulence Conf., Delft*, F. Niewstadt et al. (eds.). Kluwer Academic Press, Dordrecht

Hanjalić, K., Jakirlić, S., and Hadžić, I. 1993. Computation of oscillating turbulent flows at transitional Re-numbers. *Proc. 9th Turbulent Shear Flows*, Kyoto, 1–4

Hanjalić, K., Jakirlić, S., Stošić, N., Vasić, S., and Hadžić, 1992. Contribution to the Collaborative Testing of Turbulence Models. LSTM 352/T/92, University of Erlangen-Nürnberg, Germany

Hanjalić, K., Jones, W. P., and Launder, B. E. 1970. Some notes on energy-dissipation model of turbulence. Internal Memorandum, Imperial College of Science and Technology, London, UK

Hanjalić, K. and Launder, B. E. 1972. A Reynolds stress model of turbulence and its application to thin shear flows. *J. Fluid Mech.*, **51**, 301

Hanjalić, K. and Launder, B. E. 1976. Contribution towards a Reynolds-stress closure for low-Reynolds-number turbulence. *J. Fluid Mech.*, **74**, 593

Hanjalić, K. and Launder, B. E. 1980. Sensitizing the dissipation equation to irrotational strains. *ASME J. Fluids Eng.*, **102**, 34

Hanjalić, K., Launder, B. E., and Schiestel, R. 1980. Multiple-time-scale concepts in turbulent transport modelling. *Turbulent Shear Flows*, **2**, 36–49

Hanjalić, K. and Stošić, N. 1985. Hysteresis of turbulent stresses in wall flows subjected to periodic disturbances. *Turbulent Shear Flows*, Vol. 4, L. J. S. Bradbury et al. (eds.). Springer, Berlin, 287–300

Harlow, F. H. and Nakayama, P. I. 1968. Transport of turbulence decay rate. Rep. LA-3854, Los Alamos Science Laboratory, Los Alamos, NM

Howard, J. H. G., Patankar, S. V., and Bordynuik, R. M. 1980. Flow prediction in rotating ducts using coriolis-modified turbulence models. *ASME J. Fluids Eng.*, **102**, 456–461

Hogg, S. I. and Leschziner, M. A. 1989. Computation of highly swirling confined flow with a Reynolds stress turbulence model. *AIAA J.*, **27**, 57–63. Also Thermofluids Rept. TFD/86/10 UMIST, University of Manchester, Manchester, UK

Humphrey, J. A. C., Whitelaw, J. H., and Yee, G. 1981. Turbulent flow in a square duct with strong curvature. *J. Fluid Mech.*, **103**, 443–463

Iacovides, H. and Launder, B. E. 1984a. *Proc. 1st UK National Heat Transfer Conf.*, Inst. Chem. Eng. Symp. Series No. 86, Vol. II, 1097

Iacovides, H. and Launder, B. E. 1984b. PSL—An economic approach to the numerical analysis of near-wall, elliptic flow. *ASME J. Fluids Eng.*, **106**, 241

Iacovides, H. and Launder, B. E. 1985. ASM predictions of turbulent momentum and heat transfer in coils and U-Bends. *Proc. 4th Int. Conf. Numer. Meth. Laminar and Turbulent Flows*, Swansea

Jakirlić, S., Hanjalić, K., and Durst, F. 1993. Computation of accelerating 3-D turbulent boundary layers by a second moment closure. *Proc. 5th Int. Symp. Refined Flow Modeling and Turbulence Measurements*. ENPC Press, Paris, 57–64

Jones, W. P. and Launder, B. E. 1972. The prediction of laminarization with a two-equation model of turbulence. *Int. J. Heat Mass Transfer*, **15**, 301–314

Jones, W. P. and Musonge, P. 1983. Modeling of scalar transport in homogeneous turbulent flows. *Proc. 4th Turbulent Shear Flows Symp.*, Karlsruhe, 17.18

Jovanović, J., Ye, Q.-Y., and Durst, F. 1992. Refinement of the equation for the determination of turbulent micro-scale. LSTM Rept. 349/T/92, University of Erlangen-Nürnberg, Germany

Kolmogorov, A. N. 1941. On degeneration of isotropic turbulence in an incompressible viscous liquid. *Dokl. Akad. Nauk SSSR*, **31** (6)

Kebede, W., Launder, B. E., and Younis, B. A. 1985. Large amplitude periodic flow: a second-moment closure study. *Fifth Turbulent Shear Flows Symp.*, Cornell, Ithaca, NY

Kim, J., Moin, P., and Moser, R. 1987. Turbulence statistics in fully developed channel flow at low Reynolds number. *J. Fluid Mech.*, **177**, 133

Kline, S. J., Cantwell, B. J., and Lilley, G. M. (Eds.). 1981. *Proc. 1980–81 AFOSR-HTTM-Stanford Conf. on Complex Turbulent Flows*, Stanford University, Stanford, CA

Lai, Y. G. and So, R. M. C. 1990. On near-wall turbulent flow modelling. *J. Fluid Mech.*, **221**, 641

Lam, C. K. and Bremhorst, K. 1978. Modified form of the k–ε model for predicting wall turbulence. Report 3/78, Mechanical Engineering Dept., University of Queensland

Lakshminarayana, B. 1986. Turbulence modelling for complex shear flows. *AIAA J.*, **24** (12), 1900

Launder, B. E. 1975. On the effects of a gravitational field on the turbulent transport of heat and momentum. *J. Fluid Mech.*, **67**, 569–581

Launder, B. E. 1976. Heat and mass transport. *Topics in Applied Physics*, P. Bradshaw (ed.). Springer, Berlin, 232–287

Launder, B. E. 1990. Phenomenological modelling: present and future. *Proc. Whither Turbulence Workshop*, J. Lumley (ed.). *Lecture Notes in Physics*, **357**, 439

Launder, B. E., Reece, G. J., and Rodi, W. 1975. Progress in the development of a Re-stress turbulence closure. *J. Fluid Mech.*, **68**, 537

Launder, B. E. and Reynolds, W. C. 1982. Asymptotic near-wall stress dissipation rates in a turbulent flow. *Phys. Fluids*, **26**, 1157

Launder, B. E. and Sharma, B. I. 1974. Application of energy-dissipation model of turbulence to the calculation of flow near a spinning disc. *Lett. Heat Mass Transfer*, **1**, 131

Launder, B. E. and Shima, N. 1989. Second-moment closure for the near-wall sublayer: development and application. *AIAA J.*, **27** (10), 1319

Launder, B. E., Tselepidakis, D. P., and Younis, B. A. 1987. A second-moment closure study of rotating channel flow. *J. Fluid Mech.*, **183**, 63–75

Launder, B. E. and Tselepidakis, D. P. 1991. Progress and paradoxes in modelling near-wall turbulence. *Eighth Symp. Turbulent Shear Flows*, Munich, 29–1. Also in *Turbulent Shear Flows* (Vol. 8, pp. 81–96), Springer, Berlin, 1993

Lee, M. J. and Reynolds, W. C. 1985. On the structure of homogeneous turbulence. *Fifth Symp. Turb. Shear Flows*, Cornell, Ithaca, NY, 17–7

Leschziner, M. A. 1989. Modelling turbulent recirculating flows by finite-volume methods—current status and future directions. *Int. J. Heat Fluid Flow*, **10** (3), 186–202

Liandrat, J., Aupoix, B., Cousteix, J. 1987. Calculation of longitudinal vortices imbedded in a turbulent boundary layer. *Turbulent Shear Flows* (Vol. 5), Springer, Berlin, 253

Lilek, Ž., Nadarajah, S., Perić, M., Tindal, M. J., and Yianneskis, M. 1991. Measurements and simulation of the flow around a popet valve. *Eighth Symp. Turbulent Shear Flows*, Munich, 13–2

Lin, A. and Wolfshtein, M. 1979. *Turbulent Shear Flows* (Vol. 1). Springer Verlag, Heidelberg, 327

Lumley, J. L. 1975. Prediction methods for turbulent flow. Introduction Lecture Series 76, von Karman Institute, Rhode-St.-Genese, Belgium

Lumley, J. L. 1978. Computational modelling of turbulent flows. *Adv. Appl. Mech.*, **18**, 123

Lumley, J. L. and Khajeh Nouri, B. 1974. Computational modelling of turbulent transport. *Adv. Geophys.*, **18**, 169

Lumley, J. L. and Newman, G. R. 1977. Return to isotropy of homogeneous turbulence. *J. Fluid Mech.*, **82**, 161–178

Magnaudet, J. 1992. The modelling of inhomogeneous turbulence in the absence of mean velocity gradients. *Fourth Eur. Turbulence Conf.*, Delft, The Netherlands

Mansour, N. N., Kim, J., and Moin, P. 1988. Reynolds-stress and dissipation-rate budgets in a turbulent channel flow. *J. Fluid Mech.*, **194**, 15

Mathieu, J. and Jeandel, D. 1984. *La Simulation des Modeles de Turbulence et Leurs Applications*, Vol. I. Eyrolles, Paris

Mellor, G. L. and Herring, H. J. 1973. A survey of the mean turbulent field closure models. *AIAA J.*, **11** (5), 590

Michelassi, V., Rodi, W., and Scheuerer, G. 1991. Testing a low Reynolds number k–ε turbulence model based on direct simulation data. *Symp. Turb. Shear Flows*, Munich, 26–5

Mjolsness, R. 1979. Systematic modeling rules for Reynolds stress closures in free shear layers. *Proc. ASME Symp. Turbulent Boundary Layers*. ASME, Niagara Falls, NY, 139–144

Monin, A. S. and Yaglom, A. M. 1975. *Statistical Fluid Mechanics*, J. Lumley (ed.). MIT Press, Cambridge, MA

Moser, R. D. and Moin, P. 1984. Direct numerical simulations of curved channel flow. NASA Rep. TM85974

Nagano, Y. and Tagawa, M. 1991. Turbulence models for triple velocity and scalar correlations. *Turbulent Shear Flows* (Vol. 7). Springer, Berlin, 63–78

Nallasamy, M. 1987. Turbulence models and their application to prediction of internal flows: a review. *Comp. Fluids*, **15** (2), 151

Obi, S. 1991. Berechnung komplexer turbulenter Strömungen mit einem Reynolds-Spannungs-Modell. Doctoral thesis, University of Erlangen, Nürnberg, Germany

Obi, S., Peric, M., and Scheuerer, G. 1989. A finite-volume calculation procedure for turbulent flows with second-order closure and colocated variable arrangement. *Seventh Symp. Turbulent Shear Flows*, Stanford University, Stanford, CA, 17.4

Orszag, S. A., Yakhot, V., Flannery, W. S., Boysan, F., Choudhury, D., Maruzewski, J., and Patel, B. 1993. Renormalization group modeling and turbulence simulation. *Proc. Near-Wall Turbulent Flows*, So R. M. C. et al. (ed.). Elsevier, Amsterdam, 1031–1046

Patankar, S. V. and Spalding, D. B. 1967. *Heat and Mass Transfer in Boundary Layers*, 1st edition. Morgan-Grampian, London

Patel, V. C., Rodi, W., and Scheuerer, G. 1985. A review and evaluation of turbulence models for near-wall and law-Reynolds-number flows. *AIAA J.*, **23**, 1320

Pope, S. B. 1978. An explanation of the turbulent round-jet anomaly. *AIAA J.*, **16**, 279

Pouagare, M. and Lakshminarayana, B. 1986. Computation and turbulence closure models for shear flows in rotating curved bodies. *J. Propulsion Power*, **2**, 289–290

Prud'homme, M. and Elghobashi, S. 1983. Prediction of wall-bounded flows with an improved version of a Reynolds stress model. *Proc. 4th Turbulent Shear Flows Symp.*, 1.7, Karlsruhe

Reynolds, W. C. 1976. Computation of turbulent flows. *Annu. Rev. Fluid Mech.*, **8**, 183

Reynolds, W. C. 1984. Physical and analytical foundations, concepts and new directions in turbulence modeling and simulation. In B. E. Launder, W. C. Reynolds, and W. Rodi, *Turbulence Models and their Applications*, (Vol. II). Eyrolles, Paris

Rodi, W. 1982. Examples of turbulence models for incompressible flows. *AIAA J.*, **20** (7), 872–879

Rodi, W. 1986. Turbulence modelling for incompressible flows—A report on the Euromech 180 Colloquium. *Physico Chem. Hydr.*, **7** (5/6), 207

Rodi, W. 1988. Recent developments in turbulence modelling. *Proc. 3rd Int. Symp. Refined Flow Modeling and Turbulence Measurements*, Universal Academy Press, Tokyo

Rodi, W. and Mansour, N. N. 1993. Low Reynolds number k–ε modelling with the aid of direct simulation data. *J. Fluid Mech.*, **250**, 509–524

Rodi, W. and Spalding, D. B. 1970. A two-parameter model of turbulence and its application to free jets. *Wärme-und Stoffübertragung* B.3, 85–95

Rotta, J. C. 1951. Statistische Theorie nichthomogener Turbulenz. *Z. Phys.*, **129**, 547

Rotta, J. C. 1984. Experience of second order turbulence flow closure models. *Euromech Colloquium 180*, Karlsruhe. Also in *Zeitschrift für Flugwissenschaften und Weltraumforschung*, **10** (1986), 401–407

Savill, A. M. 1993. Some recent progress in the turbulence modelling of by-pass transition. *Proc. Int. Conf. on Near-Wall Turbulent Flows*, Tempe, AZ

Schiestel, R. 1983. Multiple-scale concept in turbulence modelling. *J. Mech. Theor. Applique*, **2** (3), 417

Schumann, U. 1977. Realizability of Reynolds-stress turbulence models. *Phys. Fluids*, **20**, 721–725

Schwarz, W. R. and Bradshaw, P. 1992. Three-dimensional turbulent boundary layer in a 30 degree bend: experiment and modelling. Stanford Rep. MD-61, Stanford University, Stanford, CA. Also Term-by-term tests of stress-transport turbulence models in a three-dimensional boundary layer. *Phys. Fluids*, to appear

Shih, T. H., Chen, J. Y. and Lumley, J. L. 1992. Second-order modeling of boundary-free turbulent shear flows. *AIAA J.*, **30** (6), 1553

Shih, T. H. and Lumley, J. L. 1985. Modelling of pressure correlation terms in Reynolds stress and scalar flux equations. Rep. FDA-85-3, Sibley School of Mechanical and Aerospace Engineering, Cornell University, Ithaca, NY

Shih, T. H. and Lumley, J. L. 1992. A critical comparison of second order closures with direct numerical simulations of homogeneous turbulence. *Fourth Eur. Turbulence Conf.*, Delft, The Netherlands

Shima, N. 1988. A Reynolds-stress model for near-wall and low-Reynolds-number regions. *ASME J. Fluids Eng.*, **110**, 38

Shima, N. 1991. Prediction of three-dimensional turbulent boundary layers using a second-moment closure. *Eighth Symp. Turbulent Shear Flows*, Munich, 8–2

Shima, N. 1993. Prediction of turbulent boundary layers with a second moment closure: Part I—Effects of periodic pressure gradient, wall transpiration and free stream turbulence; Part II—Effects of streamline curvature and spanwise rotation. *J. Fluid Eng.*, **115**, 56–69

Shir, C. C. 1973. A preliminary study of atmospheric turbulent flow in the idealized planetary boundary layer. *J. Atmospher. Sci.*, **30**, 1327

Smith, L. M. and Reynolds, W. C. 1992. On the Yakhot–Orszag renormalization group method for deriving turbulence statistics and models. *Phys. Fluids A*, **4** (2), 364–390

So, R. M. C., Lai, Y. G. and Zhang, H. S. 1991. Second-order near-wall turbulence closures: a review. *AIAA J.*, **29** (11), 1819

So, R. M. C., Zhang, H. S., and Speziale, C. G. 1991. Near-wall modeling of the dissipation rate equation. *AIAA J.*, **29** (12), 2069–2076

Spalart, P. L. 1986. Numerical study of sink-flow boundary layers. *J. Fluid Mech.*, **172**, 307

Spalart, P. L. 1988. Direct simulation of turbulent boundary layer up to $Re_\theta = 1400$. *J. Fluid Mech.*, **187**, 61

Speziale, C. G. 1987. On nonlinear k–L and k–ε models of turbulence. *J. Fluid Mech.*, **178**, 459–475

Speziale, C. G. 1991. Analytical methods for the development of Reynolds-stress closures in turbulence. *Annu. Rev. Fluid Mech.*, **23**, 107

Speziale, C. G., Abid, R., and Anderson, E. C. 1992. Critical evaluation of two-equation models for near-wall turbulence. *AIAA J.*, **30** (2), 324

Speziale, C. G., Sarkar, S., and Gatski, T. B. 1991. Modelling the pressure-strain correlation of turbulence: an invariant dynamical systems approach. *J. Fluid Mech.*, **227**, 245

Speziale, C. G., Rishi, R., and Gatski, T. B. 1990. Modeling the dissipation rate in rotating turbulent flows. *Studies in Turbulence*, T. B. Gatski et al. (eds.). Springer, Berlin, 129–142

Speziale, C. G., So, R. M., and Younis, B. A. 1993. On the prediction of turbulent secondary flows. *Proc. Near-Wall Turbulent Flows*, R. M. C. So et al. (eds.). Elsevier, Amsterdam, 105–114

Tavoularis, S. and Corrsin, S. C. 1981. Experiments in nearly homogeneous turbulent shear flow with a uniform mean temperature gradient, Pt. 1. *J. Fluid Mech.*, **104**, 311–367

Tagawa, M., Nagano, Y., and Tsuji, T. 1991. Turbulence model for the dissipation components of Reynolds stresses. *Proc. 8th Symp. Turbulent Shear Flows*, München, 29–3

Thangam, S., Abid, R., and Speziale, C. G. 1992. Application of a new k–τ model to near wall turbulent flows. *AIAA J.*, **30** (2), 552

Thangam, S. and Speziale, C. G. 1992. Turbulent flow past a backward-facing step: a critical evaluation of two-equation models. *AIAA J.*, **30** (5), 1314

Weinstock, J. 1982. Theory of pressure-strain-rate correlation for Reynolds-stress turbulence closures, Pt. 1. *J. Fluid Mech.*, **105**, 369

Weinstock, J. and Burk, S. 1985. Theoretical pressure-strain term: resistance to large anisotropies of stress and dissipation. *Proc. 5th Symp. Turb. Shear Flows*, Cornell, Ithaca, NY, 12.13

Wilcox, D. C. 1988. Reassessment of the scale-determining equation for advanced turbulence models. *AIAA J.*, **26**, 1299

Wilcox, D. and Rubesin, M. 1980. Progress in turbulence modelling for complex flow fields including effects of compressibility. NASA Technical Paper 1517

Wilcox, D. C. and Traci, R. M. 1976. A complete model of turbulence. AIAA Paper No. 76-351, San Diego, CA

Wolfstein, M., Naot, D., and Lin, A. 1975. Models in turbulence. *Topics in Transport Phenomena*, C. Gutfinger (ed.), Hemisphere, Washington, DC

Yakhot, V. and Orszag, S. A. 1986. Renormalization group analysis of turbulence. I. Basic Theory. *J. Sci. Computing*, **1** (1), 3

Yakhot, V., Orszag, S. A., Thangam, S., Gatski, T. B., and Speziale, C. G. 1992. Development of turbulence model for shear flows by double expansion technique. *Phys. Fluids A*, **4** (7), 1510

Yakhot, V. and Smith, L. M. 1992. The renormalization group, the ε-expansion and derivation of turbulence models. *J. Sci. Computing*, **3**, 35

Yoshizawa, A. 1988. A statistically derived system of equations for turbulent shear flows. *Phys. Fluids*, **28** (1), 59

Yoshizawa, A. and Liou, Meng-Sing. 1990. Effects of mean velocity shear on the distribution of the dissipation rate and turbulent kinetic energy. *Studies in Turbulence*, T. B. Gatski et al. (eds.). Springer, Berlin, 81–90